On Equivalence of Pencils from Discrete-time and Continuous-time Control

Hongguo Xu*

Abstract

We study two matrix pencils that arise, respectively, in discrete-time and continuous-time optimal and robust control. We introduce a one-to-one transformation between these two pencils. We show that for the pencils under the transformation, their regularity is preserved and their eigenvalues and deflating subspaces are equivalently related. The eigen-structures of the pencils under consideration have strong connections with the associated control problems. Our result may be applied to connect the discrete-time and continuous-time control problems and eventually lead to a unified treatment of these two types of control problems.

Keywords. D-type pencil, C-type pencil, eigen-structure, eigenvalue, deflating subspace, spectral subspace, regularity, discrete-time control, continuous-time control.
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1 Introduction

We consider two matrix pencils with special block structures. The first pencil is

$$\lambda\mathcal{E}_D - \mathcal{A}_D = \lambda \begin{bmatrix} 0 & F \\ -G^* & 0 \end{bmatrix} - \begin{bmatrix} 0 & G \\ -F^* & D \end{bmatrix}, \qquad (1)$$

where $F, G, \in \mathbb{C}^{n,m}$, and $D \in \mathbb{C}^{m,m}$ is Hermitian. The second pencil is

$$\lambda\mathcal{E}_C - \mathcal{A}_C = \lambda \begin{bmatrix} 0 & \tilde{F} \\ -\tilde{F}^* & 0 \end{bmatrix} - \begin{bmatrix} 0 & \tilde{G} \\ \tilde{G}^* & \tilde{D} \end{bmatrix}, \qquad (2)$$

[1]Department of Mathematics, University of Kansas, Lawrence Kansas 66045, USA. `xu@math.ku.edu`. Partially supported by *National Science Foundation grants 0314427 and the University of Kansas General Research Fund 2301717*

where $\tilde{F}, \tilde{G} \in \mathbb{C}^{n,m}$, and $\tilde{D} \in \mathbb{C}^{m,m}$ is Hermitian. Both pencils play the central role in optimal and robust control, see, e.g., [11, 14, 7, 18]. The pencils (1) and (2) arise in the discrete-time and continuous-time optimal and robust control, respectively. For instance, the discrete-time linear quadratic optimal control problem,

$$\begin{array}{ll} \min\limits_{u_k} \frac{1}{2} \sum\limits_{k=0}^{\infty} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \\ \text{subject to} \quad Ex_{k+1} = Ax_k + Bu_k \qquad x_0 = x^0 \end{array}$$

with $Q^* = Q$, $R^* = R$, can be related to the eigenvalue problem of the pencil

$$\lambda \left[\begin{array}{c|cc} 0 & E & 0 \\ \hline -A^* & 0 & 0 \\ -B^* & 0 & 0 \end{array}\right] - \left[\begin{array}{c|cc} 0 & A & B \\ \hline -E^* & Q & S \\ 0 & S^* & R \end{array}\right], \tag{3}$$

see, e.g., [13, 15, 1, 2, 11, 7]. The continuous-time linear quadratic optimal problem,

$$\begin{array}{ll} \min\limits_{u} \frac{1}{2} \int_0^{\infty} \begin{bmatrix} x \\ u \end{bmatrix}^* \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^* & \tilde{R} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt \\ \text{subject to} \quad \tilde{E}\dot{x} = \tilde{A}x + \tilde{B}u \qquad x(0) = x^0 \end{array}$$

with $\tilde{Q}^* = \tilde{Q}$, $\tilde{R}^* = \tilde{R}$, can be related to the eigenvalue problem of the pencil

$$\lambda \left[\begin{array}{c|cc} 0 & \tilde{E} & 0 \\ \hline -\tilde{E}^* & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{c|cc} 0 & \tilde{A} & \tilde{B} \\ \hline \tilde{A}^* & \tilde{Q} & \tilde{S} \\ \tilde{B}^* & \tilde{S}^* & \tilde{R} \end{array}\right], \tag{4}$$

see, e.g., [8, 15, 11, 7]. With the block forms as indicated the pencils (3) and (4) have the same forms as (1) and (2), respectively. For this reason we call $\lambda\mathcal{E}_D - \mathcal{A}_D$ of the form (1) the *D-type* pencil and $\lambda\mathcal{E}_C - \mathcal{A}_C$ of the form (2) the *C-type* pencil.

The pencils $\lambda\mathcal{E}_D - \mathcal{A}_D$ and $\lambda\mathcal{E}_C - \mathcal{A}_C$ have many properties that are similar to each other. For instance, $\lambda\mathcal{E}_D - \mathcal{A}_D$ is determined by the matrix triplet (F, G, D) and $\lambda\mathcal{E}_C - \mathcal{A}_C$ is determined by the same type of matrix triplet $(\tilde{F}, \tilde{G}, \tilde{D})$. The eigenvalues of $\lambda\mathcal{E}_D - \mathcal{A}_D$ are in pairs $(\lambda, \bar{\lambda}^{-1})$, i.e., they are symmetric about the unit circle, whereas the eigenvalues of $\lambda\mathcal{E}_C - \mathcal{A}_C$ are in pairs $(\lambda, -\bar{\lambda})$, i.e., they are symmetric about the imaginary axis, e.g., [9, 11]. These similarities lead to the investigation on equivalence transformations between the D-type and C-type pencils. The ultimate goal is to find such a transformation that also connects the underlying discrete-time and continuous-time control problems so that both problems can be treated in a unified way. One popular candidate is the Cayley transformation $\mathbf{c} : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, defined by

$$\mu = \mathbf{c}(\lambda) = (\lambda - 1)(\lambda + 1)^{-1}.$$

Its generalization in the space of matrix pairs (which we still call the Cayley transformation and denote by $\mathbf{c}$) is

$$(\mathcal{F}, \mathcal{B}) = \mathbf{c}(\mathcal{E}, \mathcal{A}) = (\mathcal{A} + \mathcal{E}, \mathcal{A} - \mathcal{E}),$$

see, e.g., [9, 11, 7]. Let

$$(\tilde{\mathcal{E}}, \tilde{\mathcal{A}}) = \mathbf{c}(\mathcal{E}_D, \mathcal{A}_C),$$

where $\mathcal{E}_D, \mathcal{A}_D$ are from the D-type pencil $\lambda\mathcal{E}_D - \mathcal{A}_D$. Then each eigenvalue pair $(\lambda, \bar{\lambda}^{-1})$ of $\lambda\mathcal{E}_D - \mathcal{A}_D$ is transformed to the eigenvalue pair $(\mu, -\bar{\mu})$ of $\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ with $\mu = \mathbf{c}(\lambda)$ (also $-\bar{\mu} = \mathbf{c}(\bar{\lambda}^{-1})$). So the eigenvalues of $\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ share the same symmetric pattern that the eigenvalues of $\lambda\mathcal{E}_C - \mathcal{A}_C$ have. Unfortunately, $\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ does not have the same block structure as $\lambda\mathcal{E}_C - \mathcal{A}_C$, and it can not be put into the continuous-time control setting. One possible way to remedy this is as follows. First, reduce $\lambda\mathcal{E}_D - \mathcal{A}_D$ to a so-called symplectic matrix by deflating the subpencil associated with the eigenvalue infinity. Then apply the Cayley transformation to the symplectic matrix to obtain a Hamiltonian matrix that can be viewed as the one reduced from a C-type pencil, e.g., [9, 11, 7, 12]. However, new problems arise with this approach. Firstly, a successful reduction from $\lambda\mathcal{E}_D - \mathcal{A}_D$ to a symplectic matrix requires the nonsingularity of certain matrices associated with the blocks in $\mathcal{E}_D$ and $\mathcal{A}_D$. This may not always hold, e.g., [4, 11]. Secondly, even if the nonsingularity conditions hold, with the presence of matrix inversions the resulting Hamiltonian matrix is complicated and may be still hard to interpret. Also, explicit matrix inversion may cause sever numerical instability for numerical computations (see, e.g., [15]).

In this paper we will connect the D-type pencil and the C-type pencil directly by using the simple one-to-one transformation

$$(\mathcal{E}_C, \mathcal{A}_C) = \mathbf{t}(\mathcal{E}_D, \mathcal{A}_D)$$

described in the following diagram:

$$
\begin{aligned}
\lambda\mathcal{E}_D - \mathcal{A}_D &= \lambda\begin{bmatrix} 0 & F \\ -G^* & 0 \end{bmatrix} - \begin{bmatrix} 0 & G \\ -F^* & D \end{bmatrix} \\
&\qquad\qquad \mathbf{c} \downarrow\uparrow \mathbf{c}^{-1} \\
\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}} &= \lambda\begin{bmatrix} 0 & G+F \\ -(G+F)^* & D \end{bmatrix} - \begin{bmatrix} 0 & G-F \\ (G-F)^* & D \end{bmatrix} \\
&\qquad \text{Drop } D \text{ from } \tilde{\mathcal{E}} \downarrow\uparrow \text{ add } D \text{ to } \tilde{\mathcal{E}} \\
\lambda\mathcal{E}_C - \mathcal{A}_C &= \lambda\begin{bmatrix} 0 & G+F \\ -(G+F)^* & 0 \end{bmatrix} - \begin{bmatrix} 0 & G-F \\ (G-F)^* & D \end{bmatrix} \\
&=: \lambda\begin{bmatrix} 0 & \tilde{F} \\ -\tilde{F}^* & 0 \end{bmatrix} - \begin{bmatrix} 0 & \tilde{G} \\ \tilde{G}^* & \tilde{D} \end{bmatrix}.
\end{aligned}
$$

The transformation $\mathbf{t}$ is just the Cayley transformation $\mathbf{c}$ followed by a drop/add transformation. The transformation $\mathbf{t}$ is simple, since it acts directly on the matrix triplets (F, G, D) and $(\tilde{F}, \tilde{G}, \tilde{D})$, and it only involves matrix additions and subtractions. In addition, it sets up an equivalence relation between the eigen-structures of the D-type and C-type pencils. As we will see below, under the transformation $\mathbf{t}$ the eigenvalue relation between $\lambda\mathcal{E}_D - \mathcal{A}_D$ and $\lambda\mathcal{E}_C - \mathcal{A}_C$ can be described by the Cayley transformation $\mathbf{c}$. The equivalence relation between the deflating subspaces of $\lambda\mathcal{E}_D - \mathcal{A}_D$ and $\lambda\mathcal{E}_C - \mathcal{A}_C$ can be formulated explicitly by the corresponding bases.

The paper is organized as follows. Section 2 gives some basic definitions and properties about the eigen-structure of a general matrix pencil. Section 3 contains the eigen-structure properties of the D-type and C-type pencils, and some well-known properties of the Cayley transformation. Section 4 introduces the transformation $\mathbf{t}$ and describes the relations about the eigenvalues and the deflating subspaces of the D-type and C-type pencils under the transformation $\mathbf{t}$. In the same section a generalized transformation is also introduced and its behavior is discussed. The conclusions are given in Section 5.

Throughout the paper, $\mathbb{C}, \mathbb{C}^k, \mathbb{C}^{p,q}$ are the sets of complex numbers, column vectors of dimension k, and $p \times q$ matrices, respectively. X^* is the complex conjugate transpose of matrix X. $X^{-*} = (X^{-1})^*$. $\operatorname{span} X$ is the subspace spanned by the columns of matrix X. If X has full column rank, X is called an *basis matrix* of $\operatorname{span} X$. $\operatorname{null} X$ is the null space of matrix X. $\operatorname{rank} X$ is the rank of matrix X. $\dim \mathcal{X}$ is the dimension of space $\mathcal{X}$. $0_{p\times q}$ is the $p \times q$ zero matrix and I_p (or simply I when the size is not important) is the $p \times p$ identity matrix.

2 Preliminaries

In this section we introduce some basic concepts and properties about the eigen-structure of a matrix pencil. Define

$$L_k(\lambda) = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \end{bmatrix}_{k\times(k+1)}, \quad N_k(\lambda) = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}_{k\times k}.$$

Theorem 1 (**Kronecker Canonical Form** [6, 5]). *Let $\mathcal{E}, \mathcal{A} \in \mathbb{C}^{p,q}$. Then there exist nonsingular matrices $\mathcal{Y} \in \mathbb{C}^{p,p}$ and $\mathcal{X} \in \mathbb{C}^{q,q}$ such that*

$$\mathcal{Y}^*(\lambda\mathcal{E} - \mathcal{A})\mathcal{X} = \operatorname{diag}(\mathcal{O}, \mathcal{S}_R, \mathcal{S}_L, \mathcal{J}, \mathcal{N}), \tag{5}$$

where

1. $\mathcal{O} = \lambda 0_{\alpha\times\beta} - 0_{\alpha\times\beta}$,
2. $\mathcal{S}_R = \operatorname{diag}(L_{\epsilon_1}(\lambda), \dots, L_{\epsilon_i}(\lambda))$,
3. $\mathcal{S}_L = \operatorname{diag}(L^T_{\delta_1}(\lambda), \dots, L^T_{\delta_j}(\lambda))$,
4. $\mathcal{J} = \lambda I - \operatorname{diag}(J_1, \dots, J_r)$ *with* $J_k = \operatorname{diag}(N_{\xi_{1,k}}(\lambda_k), \dots, N_{\xi_{s_k,k}}(\lambda_k))$ *for* $k = 1, \dots, r$, *and the scalars* $\lambda_1, \dots, \lambda_r$ *being distinct,*
5. $\mathcal{N} = \lambda \operatorname{diag}(N_{\eta_1}(0), \dots, N_{\eta_t}(0)) - I$.

Definition 2 *Let the pencil $\lambda\mathcal{E} - \mathcal{A}$ have the Kronecker canonical form (5). The distinct scalars $\lambda_1, \dots, \lambda_r$ from the block $\mathcal{J}$ are called the* finite eigenvalues *of $\lambda\mathcal{E} - \mathcal{A}$. For each finite eigenvalue λ_k, the integer s_k is its* geometric multiplicity*, the indices $\xi_{1,k}, \dots, \xi_{s_k,k}$ are its* partial multiplicities*, and the sum $\sum_{j=1}^{s_k} \xi_{j,k}$ is its* algebraic multiplicity.

If $\mathcal{N}$ in (5) has the size greater than zero, ∞ is also an eigenvalue of $\lambda\mathcal{E} - \mathcal{A}$. Its geometric multiplicity is t, its partial multiplicities are $\eta_1, \dots, \eta_t$, and its algebraic multiplicity is $\sum_{j=1}^{t} \eta_j$.

We denote by $\Lambda(\mathcal{E}, \mathcal{A})$ the set of all finite eigenvalues and the eigenvalue ∞ of $\lambda\mathcal{E} - \mathcal{A}$. For convenience, we denote $\Lambda(\mathcal{A}) = \Lambda(I, \mathcal{A})$.

Definition 3 *Consider the pencil* $\lambda\mathcal{E} - \mathcal{A}$ *with* $\mathcal{E}, \mathcal{A} \in \mathbb{C}^{p,q}$*.*

(a) *Suppose* $\infty \neq \lambda_0 \in \Lambda(\mathcal{E}, \mathcal{A})$ *with algebraic multiplicity* ℓ*.*

1. *The columns of* $U \in \mathbb{C}^{q,\ell}$ *span a* right spectral subspace *of* $\lambda\mathcal{E} - \mathcal{A}$ *corresponding to* λ_0*, if*

$$\mathcal{E}UT = \mathcal{A}U, \qquad \operatorname{rank} \mathcal{E}U = \ell$$

   for some matrix $T \in \mathbb{C}^{\ell,\ell}$ *with* $\Lambda(T) = \{\lambda_0\}$*.*

2. *The columns of* $V \in \mathbb{C}^{p,\ell}$ *span a* left spectral subspace *of* $\lambda\mathcal{E} - \mathcal{A}$ *corresponding to* λ_0*, if*

$$S^*V^*\mathcal{E} = V^*\mathcal{A}, \qquad \operatorname{rank} V^*\mathcal{E} = \ell$$

   for some matrix $S \in \mathbb{C}^{\ell,\ell}$ *with* $\Lambda(S^*) = \{\lambda_0\}$*.*

(b) *Suppose* $\infty \in \Lambda(\mathcal{E}, \mathcal{A})$ *with algebraic multiplicity* ℓ_∞*.*

1. *The columns of* $U \in \mathbb{C}^{q,\ell_\infty}$ *span a* right spectral subspace *of* $\lambda\mathcal{E} - \mathcal{A}$ *corresponding to* ∞*, if*

$$\mathcal{E}U = \mathcal{A}UT, \qquad \operatorname{rank} \mathcal{A}U = \ell_\infty$$

   for some nilpotent matrix $T \in \mathbb{C}^{\ell_\infty,\ell_\infty}$*, i.e.,* $\Lambda(T) = \{0\}$*.*

2. *The columns of* $V \in \mathbb{C}^{p,\ell_\infty}$ *span a* left spectral subspace *of* $\lambda\mathcal{E} - \mathcal{A}$ *corresponding to* ∞*, if*

$$V^*\mathcal{E} = S^*V^*\mathcal{A}, \qquad \operatorname{rank} V^*\mathcal{A} = \ell_\infty$$

   for some nilpotent matrix $S \in \mathbb{C}^{\ell_\infty,\ell_\infty}$*.*

For each eigenvalue of $\lambda\mathcal{E} - \mathcal{A}$ its spectral subspaces always exist. In general, they may not be unique. In [16] it is shown that every eigenvalue of $\lambda\mathcal{E} - \mathcal{A}$ has a unique right spectral subspace if and only if in the Kronecker canonical form the blocks $\mathcal{O}$ and $\mathcal{S}_R$ are void. Similarly, every eigenvalue has a unique left spectral subspace if and only if $\mathcal{O}$ and $\mathcal{S}_L$ are void. The uniqueness conditions can also be described in the following way.

Proposition 4 *Every eigenvalue in* $\Lambda(\mathcal{E}, \mathcal{A})$ *has a unique right spectral subspace if and only if* $\kappa\mathcal{E} - \mathcal{A}$ *has full column rank for some* $\kappa \in \mathbb{C}$*.*

Similarly, every eigenvalue in $\Lambda(\mathcal{E}, \mathcal{A})$ *has a unique left spectral subspace if and only if* $\kappa\mathcal{E} - \mathcal{A}$ *has full row rank for some* $\kappa \in \mathbb{C}$*.*

Proof. The result easily follows from the Kronecker canonical form. □

If the pencil $\lambda\mathcal{E} - \mathcal{A}$ is *regular*, i.e., $\mathcal{E}, \mathcal{A}$ are square and $\det(\kappa\mathcal{E} - \mathcal{A}) \neq 0$ for some $\kappa \in \mathbb{C}$, then from Proposition 4, for every eigenvalue in $\Lambda(\mathcal{E}, \mathcal{A})$ both its left and right spectral subspaces are unique. Due to this fact, when $\lambda\mathcal{E} - \mathcal{A}$ is regular, we denote by $\mathcal{R}_{\lambda_0}$ and $\mathcal{L}_{\lambda_0}$ the right and left spectral subspaces of $\lambda\mathcal{E} - \mathcal{A}$ corresponding to $\lambda_0 \in \Lambda(\mathcal{E}, \mathcal{A})$, respectively. Note that when $\lambda\mathcal{E} - \mathcal{A}$ is regular, its Kronecker canonical form becomes the Weierstraß form ([17]), i.e., the blocks $\mathcal{O}, \mathcal{S}_R, \mathcal{S}_L$ are void in (5).

Proposition 5 *Suppose $\lambda\mathcal{E} - \mathcal{A}$ is regular. Then $\infty \neq \lambda_0 \in \Lambda(\mathcal{E}, \mathcal{A})$ if and only if $\det(\lambda_0\mathcal{E} - \mathcal{A}) = 0$. $\infty \in \Lambda(\mathcal{E}, \mathcal{A})$ if and only if $\det \mathcal{E} = 0$.*

Proof. Omitted. □

Proposition 6 *Consider the regular pencil $\lambda\mathcal{E} - \mathcal{A}$ with $\mathcal{E}, \mathcal{A} \in \mathbb{C}^{p,p}$. Suppose $\infty \neq \lambda_0 \in \Lambda(\mathcal{E}, \mathcal{A})$ with algebraic multiplicity ℓ and suppose $U, V \in \mathbb{C}^{p,\ell}$. Then $\operatorname{span} U = \mathcal{R}_{\lambda_0}$ and $\operatorname{span} V = \mathcal{L}_{\lambda_0}$ if and only if $\det V^*\mathcal{E}U \neq 0$ and*

$$\mathcal{E}UT = \mathcal{A}U, \qquad S^*V^*\mathcal{E} = V^*\mathcal{A}$$

for some matrices $T, S \in \mathbb{C}^{\ell,\ell}$ with $\Lambda(T) = \Lambda(S^*) = \{\lambda_0\}$.

Similarly, suppose $\infty \in \Lambda(\mathcal{E}, \mathcal{A})$ with algebraic multiplicity ℓ_∞ and suppose $U, V \in \mathbb{C}^{p,\ell_\infty}$. Then $\operatorname{span} U = \mathcal{R}_\infty$ and $\operatorname{span} V = \mathcal{L}_\infty$ if and only if $\det V^*\mathcal{A}U \neq 0$ and

$$\mathcal{E}U = \mathcal{A}UT, \qquad V^*\mathcal{E} = S^*V^*\mathcal{A}$$

for some nilpotent matrices $T, S \in \mathbb{C}^{\ell_\infty,\ell_\infty}$.

Proof. The result easily follows from the Weierstraß form of $\lambda\mathcal{E} - \mathcal{A}$. □

Definition 7 *Let $\lambda\mathcal{E} - \mathcal{A}$ be a regular pencil.*

1. *If matrix U satisfies*

$$\mathcal{E}U = WS, \qquad \mathcal{A}U = WT,$$

where W has full column rank and $\lambda S - T$ is a regular subpencil, we call U a basis matrix of a right deflating subspace *of $\lambda\mathcal{E} - \mathcal{A}$ corresponding to $\lambda S - T$.*

In the special cases that (a) $S = I$, (b) $S = I$ and $\Lambda(T) = \{\lambda_0\}$, and (c) $T = I$ and S is nilpotent, we simply call U a basis matrix of a right deflating subspace of $\lambda\mathcal{E} - \mathcal{A}$ corresponding to (a) T, (b) the eigenvalue λ_0, and (c) the eigenvalue ∞, respectively.

2. *A matrix V is a basis matrix of a* left deflating subspace *of $\lambda\mathcal{E} - \mathcal{A}$ corresponding to $\lambda S - T$ if it is a basis matrix of a right deflating subspace of $\lambda\mathcal{E}^* - \mathcal{A}^*$ corresponding to $\lambda S^* - T^*$.*

   In the three special cases of $\lambda S - T$, similar names for the associated left deflating subspaces can also be introduced.

Note that when $\lambda\mathcal{E} - \mathcal{A}$ is regular, for each $\lambda_0 \in \Lambda(\mathcal{E}, \mathcal{A})$, $\mathcal{R}_{\lambda_0}$ ($\mathcal{L}_{\lambda_0}$) contains all right (left) deflating subspaces corresponding to λ_0. This also implies that $\mathcal{R}_{\lambda_0}$ ($\mathcal{L}_{\lambda_0}$) is the largest right (left) deflating subspace corresponding to λ_0.

3 Eigen-structures of D-type and C-type pencils, Cayley transformation

This section consists of two parts. The first part is about the eigen-structures of the D-type and C-type pencils. The second part is about the properties of the Cayley transformation. Hereafter, we only consider the regular D-type and C-type pencils.

3.1 Eigen-structures of D-type and C-type pencils

We start with the C-type pencil $\lambda\mathcal{E}_C - \mathcal{A}_C$ of the form (2). Obviously, $\lambda\mathcal{E}_C - \mathcal{A}_C$ is a skew-Hermitian/Hermitian pencil, i.e., $\mathcal{E}_C^* = -\mathcal{E}_C$ and $\mathcal{A}_C^* = \mathcal{A}_C$. So we just provide the well-known eigen-structure properties of a general skew-Hermitian/Hermitian pencil.

Proposition 8 *Consider the regular pencil $\lambda\mathcal{E} - \mathcal{A}$ with $\mathcal{E}, \mathcal{A} \in \mathbb{C}^{p,p}$ and $\mathcal{E}^* = -\mathcal{E}, \mathcal{A}^* = \mathcal{A}$.*

(a) *The eigenvalues of $\lambda\mathcal{E} - \mathcal{A}$ are in pairs $(\lambda_0, -\bar{\lambda}_0)$. Namely, $\lambda_0 \in \Lambda(\mathcal{E}, \mathcal{A})$ if and only if $-\bar{\lambda}_0 \in \Lambda(\mathcal{E}, \mathcal{A})$. Moreover, λ_0 and $-\bar{\lambda}_0$ have the same partial, algebraic, and geometric multiplicities.*

(b) *For every eigenvalue pair $(\lambda_0, -\bar{\lambda}_0)$, $\mathcal{R}_{\lambda_0} = \mathcal{L}_{-\bar{\lambda}_0}$ and $\mathcal{L}_{\lambda_0} = \mathcal{R}_{-\bar{\lambda}_0}$. More generally, U is a basis matrix of a right deflating subspace of $\lambda\mathcal{E} - \mathcal{A}$ corresponding to $\lambda S - T$ if and only if U is also a basis matrix of a left deflating subspace of $\lambda\mathcal{E} - \mathcal{A}$ corresponding to $\lambda(-S^*) - T^*$.*

When the pencil $\lambda\mathcal{E} - \mathcal{A}$ is real, there is a similar real version of the result.

Proof. The result easily follows from the structured Kronecker canonical form of $\lambda\mathcal{E} - \mathcal{A}$, see, e.g., [5, 3, 10]. □

The eigenvalue pairing $(\lambda_0, -\bar{\lambda}_0)$ actually does not apply to the purely imaginary eigenvalues, since then one has $\lambda_0 = -\bar{\lambda}_0$. But for such an eigenvalue λ_0, its right and left spectral subspaces are the same, i.e., $\mathcal{R}_{\lambda_0} = \mathcal{L}_{\lambda_0}$. This is also true for the eigenvalue ∞.

We now turn to the D-type pencil $\lambda\mathcal{E}_D - \mathcal{A}_D$ of the form (1). First we need a lemma.

Lemma 9 *Suppose that the pencil $\lambda\mathcal{E}_D - \mathcal{A}_D$ of the form (1) is regular and suppose $\infty \neq \lambda_0 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ with algebraic multiplicity ℓ.*

If λ_0 is also an eigenvalue of the pencil $\lambda G^* - F^*$ with algebraic multiplicity r_1, then $r_1 \leq \ell$, and there exist matrices

$$U = \begin{matrix} \\ n \\ m \end{matrix} \overset{\begin{matrix} r_1 & \ell - r_1 \end{matrix}}{\begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}} \in \mathbb{C}^{n+m,\ell}, \qquad T = \begin{matrix} \\ r_1 \\ \ell - r_1 \end{matrix} \overset{\begin{matrix} r_1 & \ell - r_1 \end{matrix}}{\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}} \in \mathbb{C}^{\ell,\ell}$$

with $\Lambda(T) = \{\lambda_0\}$, such that

$$\mathcal{E}_D U T = \mathcal{A}_D U, \qquad \operatorname{rank} \mathcal{E}_D U = \ell,$$

i.e., U_{11} and U are basis matrices of the right spectral subspaces of $\lambda G^* - F^*$ and $\lambda\mathcal{E}_D - \mathcal{A}_D$, respectively, corresponding to λ_0.

Similarly, if λ_0 is also an eigenvalue of the pencil $\lambda F - G$ with algebraic multiplicity r_2, then $r_2 \leq \ell$, and there exist matrices

$$V = \begin{matrix} \\ n \\ m \end{matrix} \overset{\begin{matrix} r_2 & \ell - r_2 \end{matrix}}{\begin{bmatrix} V_{11} & V_{12} \\ 0 & V_{22} \end{bmatrix}} \in \mathbb{C}^{n+m,\ell}, \qquad S = \begin{matrix} \\ r_2 \\ \ell - r_2 \end{matrix} \overset{\begin{matrix} r_2 & \ell - r_2 \end{matrix}}{\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}} \in \mathbb{C}^{\ell,\ell}$$

with $\Lambda(S^*) = \{\lambda_0\}$, such that

$$S^* V^* \mathcal{E}_D = V^* \mathcal{A}_D, \qquad \operatorname{rank} V^* \mathcal{E}_D = \ell,$$

i.e., V_{11} and V are basis matrices of the left spectral subspaces of $\lambda F - G$ and $\lambda\mathcal{E}_D - \mathcal{A}_D$, respectively, corresponding to λ_0.

Proof. Since $\lambda\mathcal{E}_D - \mathcal{A}_D$ is regular,

$$\det(\kappa\mathcal{E}_D - \mathcal{A}_D) = \det \begin{bmatrix} 0 & \kappa F - G \\ -(\kappa G^* - F^*) & -D \end{bmatrix} \neq 0$$

for some $\kappa \in \mathbb{C}$. It is clear that $\kappa G^* - F^*$ has full column rank. If $\lambda_0 \in \Lambda(G^*, F^*)$ with algebraic multiplicity r_1, by Proposition 4, the associated right spectral subspace of $\lambda G^* - F^*$ is unique and one can choose a basis matrix $U_{11} \in \mathbb{C}^{n,r_1}$ satisfying

$$G^* U_{11} T_{11} = F^* U_{11}, \qquad \operatorname{rank} G^* U_{11} = r_1$$

for some $T_{11} \in \mathbb{C}^{r_1,r_1}$ with $\Lambda(T_{11}) = \{\lambda_0\}$. Clearly, U_{11} also satisfies

$$\mathcal{E}_D \begin{bmatrix} U_{11} \\ 0 \end{bmatrix} T_{11} = \mathcal{A}_D \begin{bmatrix} U_{11} \\ 0 \end{bmatrix}, \qquad \operatorname{rank} \mathcal{E}_D \begin{bmatrix} U_{11} \\ 0 \end{bmatrix} = \operatorname{rank} G^* U_{11} = r_1.$$

This implies $\operatorname{span} \left[\begin{smallmatrix} U_{11} \\ 0 \end{smallmatrix}\right] \subseteq \mathcal{R}_{\lambda_0}$, where $\mathcal{R}_{\lambda_0}$ is the right spectral subspace of $\lambda \mathcal{E}_D - \mathcal{A}_D$ corresponding to λ_0. Hence $\left[\begin{smallmatrix} U_{11} \\ 0 \end{smallmatrix}\right]$ can be extended to the basis matrix U of $\mathcal{R}_{\lambda_0}$ as required.

Note that the pencil $\lambda \mathcal{E}_D^* - \mathcal{A}_D^*$ has the same block structure as $\lambda \mathcal{E}_D - \mathcal{A}_D$. So the second part can be proved in the same way. □

In the following theorem we will show the eigen-structure properties of $\lambda \mathcal{E}_D - \mathcal{A}_D$. The result is essentially from [11, Proposition 4.18]. The only improvement is the relation between the spectral subspaces corresponding to the eigenvalues 0 and ∞, respectively.

Theorem 10 *Suppose that $\lambda \mathcal{E}_D - \mathcal{A}_D$ of the form (1) is regular.*

(a) *Let $\lambda_0 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ ($\lambda_0 \neq 0, \infty$) with algebraic multiplicity ℓ. Then*

$$U = \begin{matrix} & \ell \\ \begin{matrix} n \\ m \end{matrix} & \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \end{matrix}, \qquad V = \begin{matrix} & \ell \\ \begin{matrix} n \\ m \end{matrix} & \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \end{matrix} \in \mathbb{C}^{n+m,\ell} \tag{6}$$

satisfy

$$\mathcal{E}_D U T = \mathcal{A}_D U, \qquad S^* V^* \mathcal{E}_D = V^* \mathcal{A}_D$$

for some $T, S \in \mathbb{C}^{\ell,\ell}$ with $\Lambda(T) = \Lambda(S^*) = \{\lambda_0\}$, if and only if

$$\hat{U} = \begin{bmatrix} U_1 T \\ U_2 \end{bmatrix}, \qquad \hat{V} = \begin{bmatrix} V_1 S \\ V_2 \end{bmatrix} \tag{7}$$

satisfy

$$\mathcal{E}_D \hat{V} S^{-1} = \mathcal{A}_D \hat{V}, \qquad T^{-*} \hat{U}^* \mathcal{E}_D = \hat{U}^* \mathcal{A}_D.$$

Moreover, $\det V^* \mathcal{E}_D U \neq 0$ *if and only if* $\det \hat{U}^* \mathcal{E}_D \hat{V} \neq 0$.

Consequently, the nonzero eigenvalues of $\lambda \mathcal{E}_D - \mathcal{A}_D$ are in pairs $(\lambda_0, \bar{\lambda}_0^{-1})$, and λ_0, $\bar{\lambda}_0^{-1}$ have the same partial, geometric, and algebraic multiplicities. For the matrices U, V in (6) and $\hat{U}, \hat{V}$ in (7), $\operatorname{span} U = \mathcal{R}_{\lambda_0}$ *and* $\operatorname{span} V = \mathcal{L}_{\lambda_0}$ *if and only if* $\operatorname{span} \hat{V} = \mathcal{R}_{\bar{\lambda}_0^{-1}}$ *and* $\operatorname{span} \hat{U} = \mathcal{L}_{\bar{\lambda}_0^{-1}}$.

(b) *Suppose* $0 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ *with algebraic multiplicity* ℓ_0 *and suppose also* $0 \in \Lambda(G^*, F^*)$ *and* $0 \in \Lambda(F, G)$ *with algebraic multiplicities* r_1 *and* r_2 *, respectively. (r_1, r_2 can be zero.) Then there exist matrices*

$$U = \begin{matrix} & r_1 & \ell_0 - r_1 \\ \begin{matrix} n \\ m \end{matrix} & \begin{bmatrix} U_{11} \\ 0 \end{bmatrix.} & \begin{matrix} U_{12} \\ U_{22} \end{matrix} \end{matrix}, \quad V = \begin{matrix} & r_2 & \ell_0 - r_2 \\ \begin{matrix} n \\ m \end{matrix} & \begin{matrix} V_{11} \\ 0 \end{matrix} & \begin{matrix} V_{12} \\ V_{22} \end{matrix} \end{matrix} \in \mathbb{C}^{n+m, \ell_0} \tag{8}$$

such that

$$\mathcal{E}_D U T = \mathcal{A}_D U, \quad S^* V^* \mathcal{E}_D = V^* \mathcal{A}_D, \qquad \operatorname{rank} \mathcal{E}_D U = \operatorname{rank} V^* \mathcal{E}_D = \ell_0,$$

where

$$T = \begin{matrix} & r_1 & \ell_0 - r_1 \\ \begin{matrix} r_1 \\ \ell_0 - r_1 \end{matrix} & \begin{matrix} T_{11} \\ 0 \end{matrix} & \begin{matrix} T_{12} \\ T_{22} \end{matrix} \end{matrix}, \qquad S = \begin{matrix} & r_2 & \ell_0 - r_2 \\ \begin{matrix} r_2 \\ \ell_0 - r_2 \end{matrix} & \begin{matrix} S_{11} \\ 0 \end{matrix} & \begin{matrix} S_{12} \\ S_{22} \end{matrix} \end{matrix}$$

are nilpotent, i.e., $\operatorname{span} U = \mathcal{R}_0$ *and* $\operatorname{span} V = \mathcal{L}_0$.

Define

$$\hat{U} = \begin{bmatrix} U_{11} & U_{11}T_{12} + U_{12}T_{22} \\ 0 & U_{22} \end{bmatrix}, \quad \hat{V} = \begin{bmatrix} V_{11} & V_{11}S_{12} + V_{12}S_{22} \\ 0 & V_{22} \end{bmatrix} \tag{9}$$

and

$$\hat{T} = \begin{bmatrix} T_{11} & T_{11}T_{12} \\ 0 & T_{22} \end{bmatrix}, \qquad \hat{S} = \begin{bmatrix} S_{11} & S_{11}S_{12} \\ 0 & S_{22} \end{bmatrix}.$$

Then

$$\mathcal{E}_D \hat{V} = \mathcal{A}_D \hat{V} \hat{S}, \quad \hat{U}^* \mathcal{E}_D = \hat{T}^* \hat{U}^* \mathcal{A}_D, \qquad \operatorname{rank} \mathcal{A}_D \hat{V} = \operatorname{rank} \hat{U}^* \mathcal{A}_D = \ell_0.$$

Consequently, if $0 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ *with algebraic multiplicity* ℓ_0*, then* $\infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ *and its algebraic multiplicity is greater than or equal to* ℓ_0*. If* U, V *in (8) are basis matrices of* $\mathcal{R}_0$ *and* $\mathcal{L}_0$*, respectively, then for* $\hat{V}, \hat{U}$ *in (9),* $\operatorname{span} \hat{V}$ *and* $\operatorname{span} \hat{U}$ *are* ℓ_0*-dimensional right and left deflating subspaces, respectively, corresponding to the eigenvalue* ∞*.*

Proof. The proof is given in Appendix A. □

Corollary 11 *Suppose that* $\lambda \mathcal{E}_D - \mathcal{A}_D$ *of the form (1) is regular. Let* $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ *with* $U_1 \in \mathbb{C}^{n,p}$ *and* $U_2 \in \mathbb{C}^{m,p}$*, and let* $T \in \mathbb{C}^{p,p}$ *be nonsingular. Then* U *is a basis matrix of a right (left) deflating subspace of* $\lambda \mathcal{E}_D - \mathcal{A}_D$ *corresponding to* T *if and only if* $\hat{U} := \begin{bmatrix} U_1 T \\ U_2 \end{bmatrix}$ *is a basis matrix of a left (right) deflating subspace of* $\lambda \mathcal{E}_D - \mathcal{A}_D$ *corresponding to* T^{-*}*.*

Proof. We first consider the special case when $\Lambda(T) = \{\lambda_0\}$ for some nonzero eigenvalue $\lambda_0 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$. In this case the result follows from the fact that $\operatorname{span} U \subseteq \mathcal{R}_{\lambda_0}$ and $\operatorname{span} \hat{U} \subseteq \mathcal{L}_{\bar{\lambda}_0^{-1}}$, and the relation between $\mathcal{R}_{\lambda_0}$ and $\mathcal{L}_{\bar{\lambda}_0^{-1}}$ shown in Theorem 10 (a).

Note that a general deflating subspace can be expressed as a direct sum of several deflating subspaces, each corresponding to a single eigenvalue. By applying the above result to each deflating subspace in the sum we can get the result for the general case. □

When the pencil $\lambda\mathcal{E}_D - \mathcal{A}_D$ is real, the real versions of Theorem 10 and Corollary 11 can be derived in the same way.

In Theorem 10 (a) the eigenvalue pairing $(\lambda_0, \bar{\lambda}_0^{-1})$ actually does not apply to the eigenvalues on the unit circle, since then one has $\lambda_0 = \bar{\lambda}_0^{-1}$. But for such an eigenvalue λ_0, its left and right spectral subspaces are related. Namely, U in (6) is a basis matrix of $\mathcal{R}_{\lambda_0}$ if and only if $\hat{U}$ in (7) is a basis matrix of $\mathcal{L}_{\lambda_0}$.

Theorem 10 (b) shows that the eigenvalues 0 and ∞ are also paired, but in a weak sense. This is because the algebraic multiplicity of ∞ may be bigger than that of 0, and $\mathcal{R}_0$ and $\mathcal{L}_0$ are only related to certain subspaces of $\mathcal{L}_\infty$ and $\mathcal{R}_\infty$, respectively.

Example 1 Consider the D-type pencil

$$\lambda\mathcal{E}_D - \mathcal{A}_D = \lambda \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline -1 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right] - \left[\begin{array}{c|cc} 0 & 1 & 1 \\ \hline -1 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right].$$

Let

$$U_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad V_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}; \quad U_2 = \begin{bmatrix} 0 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then one has

$$\mathcal{E}_D U_1 \cdot 0 = \mathcal{A}_D U_1 \qquad 0 \cdot V_1^* \mathcal{E}_D = V_1^* \mathcal{A}_D,$$

and

$$\mathcal{E}_D U_2 = \mathcal{A}_D U_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad V_2^* \mathcal{E}_D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^* V_2^* \mathcal{A}_D.$$

The pencil $\lambda\mathcal{E}_D - \mathcal{A}_D$ has two eigenvalues: 0 and ∞. Clearly, $\operatorname{span} U_1 = \mathcal{R}_0$, $\operatorname{span} V_1 = \mathcal{L}_0$ and $\operatorname{span} U_2 = \mathcal{R}_\infty$, $\operatorname{span} V_2 = \mathcal{L}_\infty$.

By (9) with $U = U_1$, $V = V_1$, and $T = 0$ (and $r_1 = r_2 = 0$), we get

$$\hat{U} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \qquad \hat{V} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

which are just the first column of V_2 and U_2, respectively.

The results in Theorem 10, Corollary 11, and Proposition 8 show that the eigen-structure properties of the pencils $\lambda\mathcal{E}_D - \mathcal{A}_D$ and $\lambda\mathcal{E}_C - \mathcal{A}_C$ can be addressed in parallel.

In the end of this subsection we give some necessary conditions for the regularity of the D-type and C-type pencils.

Proposition 12 *If $\lambda\mathcal{E}_D - \mathcal{A}_D \in \mathbb{C}^{n+m,n+m}$ of the form (1) is regular, then*

$$m - \operatorname{rank} D \le n \le m.$$

If $\lambda\mathcal{E}_C - \mathcal{A}_C \in \mathbb{C}^{n+m,n+m}$ of the form (2) is regular, then

$$m - \operatorname{rank} \tilde{D} \le n \le m.$$

Proof. If $\lambda\mathcal{E}_D - \mathcal{A}_D$ is regular, there is a scalar $\kappa \in \mathbb{C}$ such that

$$\det(\kappa\mathcal{E}_D - \mathcal{A}_D) = \det \begin{bmatrix} 0 & \kappa F - G \\ F^* - \kappa G^* & -D \end{bmatrix} \neq 0.$$

Then both $\kappa F - G, F - \bar{\kappa} G \in \mathbb{C}^{n,m}$ have full row rank. So we have $n \le m$.

Suppose $\operatorname{rank} D = p$. Let $D = Q \left[\begin{smallmatrix} \Sigma & 0 \\ 0 & 0 \end{smallmatrix}\right] Q^*$ be the Schur form, where $\Sigma \in \mathbb{C}^{p,p}$ is diagonal and nonsingular and Q is unitary. Denote

$$FQ = n \overset{\quad p \qquad m-p}{\begin{bmatrix} F_1 & F_2 \end{bmatrix}}, \qquad GQ = n \overset{\quad p \qquad m-p}{\begin{bmatrix} G_1 & G_2 \end{bmatrix}}.$$

We have

$$\begin{bmatrix} I_n & 0 \\ 0 & Q \end{bmatrix}^* (\kappa\mathcal{E}_D - \mathcal{A}_D) \begin{bmatrix} I_n & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} 0 & \kappa F_1 - G_1 & \kappa F_2 - G_2 \\ F_1^* - \kappa G_1^* & -\Sigma & 0 \\ F_2^* - \kappa G_2^* & 0 & 0 \end{bmatrix}$$

Then both $\kappa F_2 - G_2, F_2 - \bar{\kappa} G_2 \in \mathbb{C}^{n,m-p}$ have full column rank. So we have $m - p \le n$.

The second part can be proved in the same way. ∎

3.2 Cayley transformation

The Cayley transformation on $\mathbb{C}$ is defined by

$$\mu = \mathbf{c}(\lambda) = (\lambda - 1)(\lambda + 1)^{-1}.$$

It can be extended to a one-to-one transformation on $\mathbb{C} \cup \{\infty\}$, also denoted by $\mathbf{c}$, by defining $\mathbf{c}(-1) = \infty$ and $\mathbf{c}(\infty) = 1$. Its inverse transformation is

$$\lambda = \mathbf{c}^{-1}(\mu) = (1 + \mu)(1 - \mu)^{-1}.$$

The correspondence between λ and $\mu = \mathbf{c}(\lambda)$ is summarized in Table 1.

λ	$\lvert\lambda\rvert < 1$	$\lvert\lambda\rvert = 1$	$\lvert\lambda\rvert > 1$	1	0	-1	∞
μ	$\operatorname{Re}\mu < 0$	$\operatorname{Re}\mu = 0$	$\operatorname{Re}\mu > 0$	0	-1	∞	1

Table 1: Correspondence between λ and $\mu = \mathbf{c}(\lambda)$

The following relations are obvious.

Proposition 13 *If $\mu = \mathbf{c}(\lambda)$ then $-\bar{\mu} = \mathbf{c}\left(\bar{\lambda}^{-1}\right)$. Conversely, if $\lambda = \mathbf{c}^{-1}(\mu)$ then $\bar{\lambda}^{-1} = \mathbf{c}^{-1}(-\bar{\mu})$.*

Proof. the proof is trivial. □

The Cayley transformation can be extended to the matrix space $\mathbb{C}^{p,p}$:

$$\mathcal{B} = \mathbf{c}(\mathcal{A}) = (\mathcal{A} - I_p)(\mathcal{A} + I_p)^{-1},$$

and further to the space $\mathbb{C}^{p,q} \times \mathbb{C}^{p,q}$:

$$(\mathcal{F}, \mathcal{B}) = \mathbf{c}(\mathcal{E}, \mathcal{A}) = (\mathcal{A} + \mathcal{E}, \mathcal{A} - \mathcal{E}). \tag{10}$$

The relation between the eigen-structures of the pencils $\lambda\mathcal{F} - \mathcal{B}$ and $\lambda\mathcal{E} - \mathcal{A}$ connected by (10) is summarized below.

Proposition 14 *Consider $\lambda\mathcal{E} - \mathcal{A}$ and $\lambda\mathcal{F} - \mathcal{B}$ where $(\mathcal{F}, \mathcal{B}) = \mathbf{c}(\mathcal{E}, \mathcal{A})$.*

(a) *$\lambda\mathcal{E} - \mathcal{A}$ is regular if and only if $\lambda\mathcal{F} - \mathcal{B}$ is regular.*

(b) *$\lambda_0 \in \Lambda(\mathcal{E}, \mathcal{A})$ if and only if $\mu_0 = \mathbf{c}(\lambda_0) \in \Lambda(\mathcal{F}, \mathcal{B})$. Moreover, λ_0, μ_0 have the same partial, geometric, and algebraic multiplicities.*

(c) *Suppose $\lambda\mathcal{E} - \mathcal{A}$ is regular. For any $\lambda_0 \in \Lambda(\mathcal{E}, \mathcal{A})$, let $\mathcal{R}_{\lambda_0}$, $\mathcal{L}_{\lambda_0}$ be the associated right and left spectral subspaces of $\lambda\mathcal{E} - \mathcal{A}$, respectively, and let $\mathcal{R}_{\mu_0}$, $\mathcal{L}_{\mu_0}$ be the right and left spectral subspaces of $\lambda\mathcal{F} - \mathcal{B}$, respectively, corresponding to $\mu_0 = \mathbf{c}(\lambda_0)$. Then $\mathcal{R}_{\lambda_0} = \mathcal{R}_{\mu_0}$, $\mathcal{L}_{\lambda_0} = \mathcal{L}_{\mu_0}$.*

Proof. The result can be found in [9, 11, 12]. □

4 Equivalence relation between D-type and C-type pencils

In this section we introduce a transformation between the pencils $\lambda\mathcal{E}_D - \mathcal{A}_D$ and $\lambda\mathcal{E}_C - \mathcal{A}_C$ of the forms (1) and (2), respectively. We will also show the eigen-structure relation between these two pencils under the transformation.

The transformation, denoted by $(\mathcal{E}_C, \mathcal{A}_C) = \mathbf{t}(\mathcal{E}_D, \mathcal{A}_D)$, is defined by

$$\left(\begin{bmatrix} 0 & \tilde{F} \\ -\tilde{F}^* & 0 \end{bmatrix}, \begin{bmatrix} 0 & \tilde{G} \\ \tilde{G}^* & \tilde{D} \end{bmatrix}\right) = \mathbf{t}\left(\begin{bmatrix} 0 & F \\ -G^* & 0 \end{bmatrix}, \begin{bmatrix} 0 & G \\ -F^* & D \end{bmatrix}\right)$$
$$:= \left(\begin{bmatrix} 0 & G+F \\ -(G+F)^* & 0 \end{bmatrix}, \begin{bmatrix} 0 & G-F \\ (G-F)^* & D \end{bmatrix}\right).$$

Since $\lambda\mathcal{E}_D - \mathcal{A}_D$ and $\lambda\mathcal{E}_C - \mathcal{A}_C$ are determined by the matrix triplets (F, G, D) and $(\tilde{F}, \tilde{G}, \tilde{D})$, a compact version of the transformation can be introduced though the matrix triplets:

$$(\tilde{F}, \tilde{G}, \tilde{D}) = \mathbf{t}(F, G, D) = (G + F, G - F, D).$$

Clearly, $\mathbf{t}$ is invertible and the inverse transformation $(\mathcal{E}_D, \mathcal{A}_D) = \mathbf{t}^{-1}(\mathcal{E}_C, \mathcal{A}_C)$ can be described by the inverse of the compact version

$$(F, G, D) = \mathbf{t}^{-1}(\tilde{F}, \tilde{G}, \tilde{D}) = \left(\frac{1}{2}(\tilde{F} - \tilde{G}),\ \frac{1}{2}(\tilde{F} + \tilde{G}),\ \tilde{D}\right).$$

The transformation $\mathbf{t}$ can also be viewed as the composition of the Cayley transformation $\mathbf{c}$ and a "drop/add" transformation:

$$\lambda\mathcal{E}_D - \mathcal{A}_D = \lambda\begin{bmatrix} 0 & F \\ -G^* & 0 \end{bmatrix} - \begin{bmatrix} 0 & G \\ -F^* & D \end{bmatrix}$$

$$\mathbf{c} \downarrow\uparrow \mathbf{c}^{-1}$$

$$\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}} = \lambda\begin{bmatrix} 0 & G+F \\ -(G+F)^* & D \end{bmatrix} - \begin{bmatrix} 0 & G-F \\ (G-F)^* & D \end{bmatrix}$$

$$\text{Drop } D \text{ from } \tilde{\mathcal{E}} \downarrow\uparrow \text{ add } D \text{ to } \tilde{\mathcal{E}}$$

$$\lambda\mathcal{E}_C - \mathcal{A}_C = \lambda\begin{bmatrix} 0 & G+F \\ -(G+F)^* & 0 \end{bmatrix} - \begin{bmatrix} 0 & G-F \\ (G-F)^* & D \end{bmatrix}$$
$$=: \lambda\begin{bmatrix} 0 & \tilde{F} \\ -\tilde{F}^* & 0 \end{bmatrix} - \begin{bmatrix} 0 & \tilde{G} \\ \tilde{G}^* & \tilde{D} \end{bmatrix}.$$

Note that when $(\mathcal{E}_C, \mathcal{A}_C) = \mathbf{t}(\mathcal{E}_D, \mathcal{A}_D)$, the pencil $\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ with $(\tilde{\mathcal{E}}, \tilde{\mathcal{A}}) = \mathbf{c}(\mathcal{E}_D, \mathcal{A}_D)$ can be expressed as

$$\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}} = \lambda \begin{bmatrix} 0 & \tilde{F} \\ -\tilde{F}^* & \tilde{D} \end{bmatrix} - \begin{bmatrix} 0 & \tilde{G} \\ \tilde{G}^* & \tilde{D} \end{bmatrix}. \tag{11}$$

Our first result shows that the regularity of $\lambda\mathcal{E}_D - \mathcal{A}_D$ and $\lambda\mathcal{E}_C - \mathcal{A}_C$ is preserved under the transformation $\mathbf{t}$.

Theorem 15 *The pencil $\lambda\mathcal{E}_D - \mathcal{A}_D$ is regular if and only if $\lambda\mathcal{E}_C - \mathcal{A}_C$ with $(\mathcal{E}_C, \mathcal{A}_C) = \mathbf{t}(\mathcal{E}_D, \mathcal{A}_D)$ is regular.*

Proof. Let $(\tilde{\mathcal{E}}, \tilde{\mathcal{A}}) = \mathbf{c}(\mathcal{E}_D, \mathcal{A}_D)$. Due to Proposition 14 (a), it is sufficient to show that $\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ is regular if and only if $\lambda\mathcal{E}_C - \mathcal{A}_C$ is regular.

Suppose $\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ is regular. Then $\det(\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}}) \not\equiv 0$. This implies that $\det(\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}})$ is a nonzero polynomial of λ. So one can always choose a scalar $\kappa \neq -1$ such that $\det(\kappa\tilde{\mathcal{E}} - \tilde{\mathcal{A}}) \neq 0$. The same argument applies to $\lambda\mathcal{E}_C - \mathcal{A}_C$.

From the block form (11), for $\kappa \neq -1$ we have

$$\begin{aligned}
&\det(\kappa\tilde{\mathcal{E}} - \tilde{\mathcal{A}}) = \det \begin{bmatrix} 0 & \kappa\tilde{F} - \tilde{G} \\ -\kappa\tilde{F}^* - \tilde{G}^* & (\kappa - 1)\tilde{D} \end{bmatrix} \\
&= \det \left(\begin{bmatrix} (1-\kappa)^{-1} I_n & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} 0 & \kappa\tilde{F} - \tilde{G} \\ -\kappa\tilde{F}^* - \tilde{G}^* & -\tilde{D} \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & (1-\kappa) I_m \end{bmatrix} \right) \\
&= (1-\kappa)^{m-n} \det(\kappa\mathcal{E}_C - \mathcal{A}_C).
\end{aligned}$$

Hence, $\det(\kappa\tilde{\mathcal{E}} - \tilde{\mathcal{A}}) \neq 0$ if and only if $\det(\kappa\mathcal{E}_C - \mathcal{A}_C) \neq 0$. □

We turn to study the eigen-structure relation between the D-type and C-type pencils under the transformation $\mathbf{t}$. In order to avoid confusions, we denote by $\mathcal{R}^D_{\lambda_0}$ and $\mathcal{L}^D_{\lambda_0}$ the right and left spectral subspaces of $\lambda\mathcal{E}_D - \mathcal{A}_D$, respectively, corresponding to the eigenvalue λ_0; and by $\mathcal{R}^C_{\mu_0}$ and $\mathcal{L}^C_{\mu_0}$ the right and left spectral subspaces of $\lambda\mathcal{E}_C - \mathcal{A}_C$, respectively, corresponding to the eigenvalue μ_0.

Theorem 16 *Consider the D-type pencil $\lambda\mathcal{E}_D - \mathcal{A}_D$ of the form (1) and the C-type pencil $\lambda\mathcal{E}_C - \mathcal{A}_C$ of the form (2). Suppose that $(\mathcal{E}_C, \mathcal{A}_C) = \mathbf{t}(\mathcal{E}_D, \mathcal{A}_D)$ and $\lambda\mathcal{E}_D - \mathcal{A}_D$ (or $\lambda\mathcal{E}_C - \mathcal{A}_C$) is regular.*

(a) *Let $\lambda_0 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ ($\lambda_0 \neq -1, \infty$) with algebraic multiplicity ℓ. Then*

$$U = \begin{matrix} n \\ m \end{matrix} \overset{\ell}{\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}}, \quad V = \begin{matrix} n \\ m \end{matrix} \overset{\ell}{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}} \in \mathbb{C}^{n+m,\ell} \tag{12}$$

satisfy

$$\mathcal{E}_D U T = \mathcal{A}_D U, \quad S^* V^* \mathcal{E}_D = V^* \mathcal{A}_D$$

for some $T, S \in \mathbb{C}^{\ell,\ell}$ *with* $\Lambda(T) = \Lambda(S^*) = \{\lambda_0\}$, *if and only if*

$$\tilde{U} = \begin{bmatrix} U_1(I+T) \\ 2U_2 \end{bmatrix}, \quad \tilde{V} = \begin{bmatrix} V_1(I+S) \\ 2V_2 \end{bmatrix} \tag{13}$$

satisfy

$$\mathcal{E}_C \tilde{U} \tilde{T} = \mathcal{A}_C \tilde{U}, \quad \tilde{S}^* \tilde{V}^* \mathcal{E}_C = \tilde{V}^* \mathcal{A}_C,$$

where $\tilde{T} = \mathbf{c}(T)$, $\tilde{S} = \mathbf{c}(S)$, *and* $\Lambda(\tilde{T}) = \Lambda(\tilde{S}^*) = \{\mu_0\}$ *with* $\mu_0 = \mathbf{c}(\lambda_0)$. *Moreover,* $\det V^* \mathcal{E}_D U \neq 0$, *if and only if* $\det \tilde{V}^* \mathcal{E}_C \tilde{U} \neq 0$.

Consequently, $\lambda_0 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ *(*$\lambda_0 \neq -1, \infty$*) if and only if* $\mu_0 = \mathbf{c}(\lambda_0) \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$ *(*$\mu_0 \neq \infty, 1$*). Both* λ_0 *and* μ_0 *have the same partial, geometric, and algebraic multiplicities. For the matrices* U, V *in (12) and* $\tilde{U}, \tilde{V}$ *in (13),* $\operatorname{span} U = \mathcal{R}^D_{\lambda_0}$ *and* $\operatorname{span} V = \mathcal{L}^D_{\lambda_0}$ *if and only if* $\operatorname{span} \tilde{U} = \mathcal{R}^C_{\mu_0}$ *and* $\operatorname{span} \tilde{V} = \mathcal{L}^C_{\mu_0}$.

(b) *Suppose* $-1 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ *with algebraic multiplicity* ℓ_{-1}. *Suppose also* $-1 \in \Lambda(G^*, F^*)$ *with algebraic multiplicity* r_1. *Then there exists matrix*

$$U = \begin{matrix} & \begin{matrix} r_1 & \ell_{-1} - r_1 \end{matrix} \\ \begin{matrix} n \\ m \end{matrix} & \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} \end{matrix} \in \mathbb{C}^{n+m, \ell_{-1}}, \tag{14}$$

such that

$$\mathcal{E}_D U T = \mathcal{A}_D U, \qquad \operatorname{rank} \mathcal{E}_D U = \ell_{-1},$$

where

$$T = \begin{matrix} & \begin{matrix} r_1 & \ell_{-1} - r_1 \end{matrix} \\ \begin{matrix} r_1 \\ \ell_{-1} - r_1 \end{matrix} & \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \end{matrix} \in \mathbb{C}^{\ell_{-1}, \ell_{-1}}$$

with $\Lambda(T) = \{-1\}$, *i.e.,* $\operatorname{span} U = \mathcal{R}^D_{-1}$.

Define

$$\tilde{U} = \begin{bmatrix} 2U_{11} & U_{12}(T_{22} + I) \\ 0 & 2U_{22} \end{bmatrix} \tag{15}$$

and

$$\tilde{T} = \begin{bmatrix} (T_{11} + I)(T_{11} - I)^{-1} & (I - T_{11})^{-1} T_{12} (T_{22} + I)(T_{22} - I)^{-1} \\ 0 & (T_{22} + I)(T_{22} - I)^{-1} \end{bmatrix}.$$

Then $\tilde{T}$ is nilpotent, and $\tilde{U}$, $\tilde{T}$ satisfy

$$\mathcal{E}_C\tilde{U} = \mathcal{A}_C\tilde{U}\tilde{T}, \qquad \operatorname{rank}\mathcal{A}_C\tilde{U} = \ell_{-1}.$$

Consequently, if $-1 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ with algebraic multiplicity ℓ_{-1}, then $\infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$ and its algebraic multiplicity is greater than or equal to ℓ_{-1}. If U in (14) is a basis matrix of $\mathcal{R}^D_{-1}$, then the columns of $\tilde{U}$ in (15) span an ℓ_{-1}-dimensional (both right and left) deflating subspace of $\lambda\mathcal{E}_C - \mathcal{A}_C$ corresponding to the eigenvalue ∞.

(c) *Let ℓ_{-1}, ℓ_0, and ℓ_∞ be the algebraic multiplicities of the eigenvalues $-1, 0, \infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$, respectively; and $\tilde{\ell}_1$ and $\tilde{\ell}_\infty$ be the algebraic multiplicities of the eigenvalues $1, \infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$, respectively. Then $\ell_0 = \tilde{\ell}_1$ and*

$$\tilde{\ell}_\infty = \ell_\infty - \ell_0 + \ell_{-1}, \qquad \ell_\infty = \tilde{\ell}_\infty - \ell_{-1} + \tilde{\ell}_1.$$

More precisely, with the transformation $\mathbf{t}$, the eigenvalue $\infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$ is transformed from $-1 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ (with multiplicity ℓ_{-1}) and $\infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ (with multiplicity $\ell_\infty - \ell_0$). With the inverse transformation $\mathbf{t}^{-1}$, the eigenvalue $\infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ is transformed from $1 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$ (with multiplicity $\tilde{\ell}_1$) and $\infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$ (with multiplicity $\tilde{\ell}_\infty - \ell_{-1}$).

Proof. The proof is given in Appendix B. □

Corollary 17 *Suppose that $\lambda\mathcal{E}_D - \mathcal{A}_D$ of the form (1) and $\lambda\mathcal{E}_C - \mathcal{A}_C$ of the form (2), with $(\mathcal{E}_C, \mathcal{A}_C) = \mathbf{t}(\mathcal{E}_D, \mathcal{A}_D)$, are regular. Let $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ with $U_1 \in \mathbb{C}^{n,p}$ and $U_2 \in \mathbb{C}^{m,p}$, and let $T \in \mathbb{C}^{p,p}$ and $-1 \notin \Lambda(T)$. Then U is a basis matrix of a right (left) deflating subspace of $\lambda\mathcal{E}_D - \mathcal{A}_D$ corresponding to T if and only if $\hat{U} := \begin{bmatrix} U_1(I+T) \\ 2U_2 \end{bmatrix}$ is a basis matrix of a right (left) deflating subspace of $\lambda\mathcal{E}_C - \mathcal{A}_C$ corresponding to $\mathbf{c}(T)$.*

Proof. The result easily follows from Theorem 16 (a), by using the same argument used in the proof of Corollary 11. □

The real versions of Theorem 16 and Corollary 17 can be derived in the same way.

The eigenvalue relation between $\lambda\mathcal{E}_D - \mathcal{A}_D$ and $\lambda\mathcal{E}_C - \mathcal{A}_C$ under the transformation $\mathbf{t}$ is summarized in Table 2.

From Theorem 16 (a) and Proposition 13, the transformation $\mathbf{t}$ sets up the correspondence between the eigenvalue pairs $(\lambda_0, \bar{\lambda}_0^{-1})$ of $\lambda\mathcal{E}_D - \mathcal{A}_D$ and $(\mu_0, -\bar{\mu}_0)$ of $\lambda\mathcal{E}_C - \mathcal{A}_C$ with $\mu_0 = \mathbf{c}(\lambda_0)$ (and $-\bar{\mu}_0 = \mathbf{c}(\bar{\lambda}_0^{-1})$). This also applies

$\lambda\mathcal{E}_D - \mathcal{A}_D$	$\|\lambda\| < 1$	$\|\lambda\| = 1$	$\|\lambda\| > 1$	1	0	-1		∞
$\mathbf{t} \updownarrow \mathbf{t}^{-1}$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\swarrow\nearrow$	$\updownarrow$
$\lambda\mathcal{E}_C - \mathcal{A}_C$	$\operatorname{Re}\mu < 0$	$\operatorname{Re}\mu = 0$	$\operatorname{Re}\mu > 0$	0	-1	∞		1

Table 2: Relation between the eigenvalues of $\lambda\mathcal{E}_D - \mathcal{A}_D$ and $\lambda\mathcal{E}_C - \mathcal{A}_C$

to the weak pair $(0, \infty)$ of $\lambda\mathcal{E}_D - \mathcal{A}_D$ and the pair $(-1, 1)$ of $\lambda\mathcal{E}_C - \mathcal{A}_C$. Theorem 16 does not show the relation between $\infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ and $1 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$. However, the information about the eigenvalue $1 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$ can be easily extracted. By Proposition 8 the eigenvalues $-1, 1 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$ must be paired. This means that if $-1 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$ (or equivalently, $0 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$) then $1 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$ and it must be transformed from $\infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ (since $\mathbf{c}(\infty) = 1$). In addition, there is no need to extract the spectral subspaces $\mathcal{R}_1^C$ and $\mathcal{L}_1^C$ from the spectral subspaces corresponding to $\infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$, since by Proposition 8, $\mathcal{R}_1^C = \mathcal{L}_{-1}^C$ and $\mathcal{L}_1^C = \mathcal{R}_{-1}^C$.

Table 2 and Table 1 show that the eigenvalue relation between $\lambda\mathcal{E}_D - \mathcal{A}_D$ and $\lambda\mathcal{E}_C - \mathcal{A}_C$ under the transformation $\mathbf{t}$ is almost the same as the relation between $\lambda\mathcal{E}_D - \mathcal{A}_D$ and $\lambda\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$ under the Cayley transformation $\mathbf{c}$. The only difference is that $\mathbf{t}$ creates a direct connection between $\infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ and $\infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$. So only part of $\infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ (by counting multiplicity) is transformed to $1 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$ (to match $-1 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$). The Cayley transformation $\mathbf{c}$ transforms all $\infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ to $1 \in \Lambda(\tilde{\mathcal{E}}, \tilde{\mathcal{A}})$, which causes the mismatch between the eigenvalues $-1, 1 \in \Lambda(\tilde{\mathcal{E}}, \tilde{\mathcal{A}})$. This may explain why the Cayley transformation alone can not transform a D-type pencil to a skew-Hermitian/Hermitian pencil.

By Theorem 10, if $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ is a basis matrix of $\mathcal{R}_{\lambda_0}^D$, then $\hat{U} = \begin{bmatrix} U_1 T \\ U_2 \end{bmatrix}$ is a basis matrix of $\mathcal{L}_{\bar{\lambda}_0^{-1}}^D$. Theorem 16 shows that when $\lambda_0 \neq -1$, the sum $\tilde{U} = U + \hat{U}$ is just a basis matrix of $\mathcal{R}_{\mu_0}^C$. By Proposition 8, $\tilde{U}$ is also a basis matrix of $\mathcal{L}_{-\bar{\mu}_0}^C$. The same argument applies to V, $\hat{V}$, and $\tilde{V}$. This describes the precise relation about the right and left spectral subspaces of $\lambda_0, \bar{\lambda}_0^{-1} \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ and $\mu_0, -\bar{\mu}_0 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$.

Theorem 16 shows how to extract the eigen-structure information about $\infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$ from the information about $-1 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$. Conversely, things are still not clear. Certainly, one can always work on the pencil $\lambda\mathcal{E}_D - \mathcal{A}_D$ directly. The following result, however, shows that we can still use the information of $\lambda\mathcal{E}_C - \mathcal{A}_C$ if we only need to check if $-1 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$.

Proposition 18 *Suppose that $\lambda\mathcal{E}_D - \mathcal{A}_D$ of the form (1) and $\lambda\mathcal{E}_C - \mathcal{A}_C$ of the form (2), with $(\mathcal{E}_D, \mathcal{A}_D) = \mathbf{t}^{-1}(\mathcal{E}_C, \mathcal{A}_C)$, are regular. Suppose that Q is a* *basis matrix of* $\operatorname{null} \tilde{F}$*. Let* $\hat{D} = Q^* \tilde{D} Q$*. Then* $-1 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ *if and only if at least one of the following conditions holds.*

(i) $\operatorname{rank} \tilde{F} < n$*.*

(ii) $\hat{D}$ *is singular.*

Moreover, let $p = \operatorname{rank} \tilde{F}$ *and* $q = \operatorname{rank} \hat{D}$*. Then the geometric multiplicity of* $-1 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ *is*

$$\dim \operatorname{null}(\mathcal{E}_D + \mathcal{A}_D) = n + m - 2p - q.$$

Proof. Recall $\tilde{\mathcal{E}} = \mathcal{E}_D + \mathcal{A}_D$, where $\tilde{\mathcal{E}}$ is from the pair $(\tilde{\mathcal{E}}, \tilde{\mathcal{A}}) = \mathbf{c}(\mathcal{E}_D, \mathcal{A}_D)$ and has the block form (11). Due to the fact that $-1 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ if and only if $\operatorname{rank}(\mathcal{E}_D + \mathcal{A}_D) < n+m$, and $\dim \operatorname{null}(\mathcal{E}_D + \mathcal{A}_D) = n+m-\operatorname{rank}(\mathcal{E}_D + \mathcal{A}_D)$, it is sufficient to show that $\operatorname{rank} \tilde{\mathcal{E}} = 2p+q$, and $2p+q < n+m$ if and only if $p < n$ and/or $q < m - p$.

Let

$$\tilde{F} = W \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \tilde{Q}^*$$

be the singular value decomposition of $\tilde{F}$, where $\Sigma \in \mathbb{C}^{p,p}$ is positive diagonal, $W \in \mathbb{C}^{n,n}$ and $\tilde{Q} \in \mathbb{C}^{m,m}$ are unitary. Since the last $m-p$ columns of $\tilde{Q}$ also span the null space of $\tilde{F}$, without the loss of generality we express

$$\tilde{Q} = \begin{bmatrix} \tilde{Q}_1 & Q \end{bmatrix}.$$

Let

$$\hat{D} = V \begin{bmatrix} 0 & 0 \\ 0 & \Delta \end{bmatrix} V^*$$

be the Schur form of $\hat{D} = Q^* \tilde{D} Q$, where $\Delta \in \mathbb{C}^{q,q}$ is diagonal and nonsingular, $V \in \mathbb{C}^{m-p,m-p}$ is unitary. Define

$$\mathcal{P} = \operatorname{diag}\left(W, \begin{bmatrix} \tilde{Q}_1 & QV \end{bmatrix}\right).$$

Then by using the block form (11), we have

$$\mathcal{P}^* \tilde{\mathcal{E}} \mathcal{P} = \left[\begin{array}{cc|ccc} 0 & 0 & \Sigma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline -\Sigma & 0 & \tilde{D}_{11} & \tilde{D}_{12} & \tilde{D}_{13} \\ 0 & 0 & \tilde{D}_{12}^* & 0 & 0 \\ 0 & 0 & \tilde{D}_{13}^* & 0 & \Delta \end{array}\right].$$

Clearly, $\operatorname{rank} \tilde{\mathcal{E}} = 2p + q$. So $\tilde{\mathcal{E}}$ is nonsingular if and only if $2p + q = n + m$. Since $p \leq n$ (due to the fact $n \leq m$ given in Proposition 12) and $q \leq m - p$, one can easily verified that $2p+q = n+m$ if and only if $p = n$ and $q = m-n$. Equivalently, $\tilde{\mathcal{E}}$ is singular if and only if $p < n$ or $q < m-p$, or both. □

The reason for causing the above little trouble is because -1 happens to be the pole of the Cayley transformation $\mathbf{c}$, which introduces the relation between $-1 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ and $\infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$. One way to fix this problem is to replace the Cayley transformation $\mathbf{c}$ in $\mathbf{t}$ by the generalized Cayley transformation

$$(\mathcal{F}, \mathcal{B}) = \mathbf{c}_{\alpha,h}(\mathcal{E}, \mathcal{A}) = ((\mathcal{A} + e^{i\alpha}\mathcal{E})\mathcal{I},\ h(\mathcal{A} - e^{i\alpha}\mathcal{E})\mathcal{I}),$$

where $0 \le \alpha < 2\pi$, $h > 0$, and $\mathcal{I} = \mathrm{diag}(e^{-i\alpha}I_n, I_m)$. (Here we assume $\mathcal{E}, \mathcal{A} \in \mathbb{C}^{n+m,n+m}$.) By applying $\mathbf{c}_{\alpha,h}$ to $(\mathcal{E}_D, \mathcal{A}_D)$ followed by the same drop/add transformation, we get a parameterized transformation from the D-type pencil to the C-type pencil:

$$\begin{aligned}&(\mathcal{E}_C, \mathcal{A}_C) = \mathbf{t}_{\alpha,h}(\mathcal{E}_D, \mathcal{A}_D)\\ &= \left(\begin{bmatrix} 0 & G + e^{i\alpha}F \\ -(G + e^{i\alpha}F)^* & 0 \end{bmatrix}, \begin{bmatrix} 0 & h(G - e^{i\alpha}F) \\ h(G - e^{i\alpha}F)^* & hD \end{bmatrix} \right).\end{aligned}$$

Note that $\mathbf{t}_{\alpha,h}$ is invertible. Note also that $\mathbf{c}_{\alpha,h}$ and $\mathbf{t}_{\alpha,h}$ generalize $\mathbf{c}$ and $\mathbf{t}$, respectively, because $\mathbf{c} = \mathbf{c}_{0,1}$ and $\mathbf{t} = \mathbf{t}_{0,1}$. The eigenvalue relation between $\lambda\mathcal{E}_D - \mathcal{A}_D$ and $\lambda\mathcal{E}_C - \mathcal{A}_C$ under $\mathbf{t}_{\alpha,h}$ can be described by the scalar generalized Cayley transformation

$$\mu = \mathbf{c}_{\alpha,h}(\lambda) = h(e^{-i\alpha}\lambda - 1)(e^{-i\alpha}\lambda + 1)^{-1} = h\mathbf{c}(e^{-i\alpha}\lambda),$$

which can be decomposed into three transformations:

$$\lambda \overset{\text{Rotation}}{\longleftrightarrow} e^{-i\alpha}\lambda \overset{\text{Cayley}}{\longleftrightarrow} \mathbf{c}(e^{-i\alpha}\lambda) \overset{\text{Scaling}}{\longleftrightarrow} h\mathbf{c}(e^{-i\alpha}\lambda) = \mu.$$

The transformation $\mathbf{t}_{\alpha,h}$ has its advantages. For $(\mathcal{E}_C, \mathcal{A}_C) = \mathbf{t}_{\alpha,h}(\mathcal{E}_D, \mathcal{A}_D)$, we are able to select an α such that $e^{-i\alpha}\lambda \ne -1$ for all $\lambda \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$. Then no finite eigenvalues of $\lambda\mathcal{E}_D - \mathcal{A}_D$ will be transformed to $\infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$. Therefore, the problem happened to $\mathbf{t}$ will not occur. Similarly, for $(\mathcal{E}_D, \mathcal{A}_D) = \mathbf{t}_{\alpha,h}^{-1}(\mathcal{E}_C, \mathcal{A}_C)$, we are able to select an $h > 0$ such that $h^{-1}\mu \ne 1, -1$ for all $\mu \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$. Then no finite eigenvalues of $\lambda\mathcal{E}_C - \mathcal{A}_C$ will be transformed to $\infty, 0 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$. Under the transformation $\mathbf{t}_{\alpha,h}$, the eigen-structure relation between $\lambda\mathcal{E}_D - \mathcal{A}_D$ and $\lambda\mathcal{E}_C - \mathcal{A}_C$ is similar to that under $\mathbf{t}$ given in Theorem 16 and Corollary 17, and can be proved in the same way. However, there is still a problem for real pencils. When $\lambda\mathcal{E}_D - \mathcal{A}_D$ is real, in order to keep $\lambda\mathcal{E}_C - \mathcal{A}_C$ real one has to select one of the transformations $\mathbf{t}_{0,h}$ and $\mathbf{t}_{\pi,h}$. If it happens that both -1 and 1 are eigenvalues of $\lambda\mathcal{E}_D - \mathcal{A}_D$, then no one is able to remove the problem discussed above.

Finally, we give two examples to illustrate the behavior of the eigenvalue ∞ and its associated deflating subspaces under the transformation $\mathbf{t}$.

Example 2 Consider the D-type pencil

$$\lambda\mathcal{E}_D - \mathcal{A}_D = \lambda \left[\begin{array}{c|cc} 0 & -1 & 0 \\ \hline -1 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right] - \left[\begin{array}{c|cc} 0 & 1 & 1 \\ \hline 1 & 0 & a \\ 0 & a & b \end{array}\right],$$

where $a \neq b$. Let

$$U_1 = \begin{bmatrix} a & b-a \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad V_1 = \begin{bmatrix} a & -b \\ -1 & 0 \\ 0 & 1 \end{bmatrix}; \qquad U_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Then

$$\begin{aligned} &\mathcal{E}_D U_1 \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} = \mathcal{A}_D U_1, \quad \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}^* V_1^* \mathcal{E}_D = V_1^* \mathcal{A}_D; \\ &\mathcal{E}_D U_2 = \mathcal{A}_D U_2 \cdot 0, \quad V_2^* \mathcal{E}_D = 0 \cdot V_2^* \mathcal{A}_D. \end{aligned}$$

So $\lambda\mathcal{E}_D - \mathcal{A}_D$ has the eigenvalues -1 and ∞.

Under the transformation $\mathbf{t}$, the corresponding C-type pencil is

$$\lambda\mathcal{E}_C - \mathcal{A}_C = \lambda \left[\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right] - \left[\begin{array}{c|cc} 0 & 2 & 1 \\ \hline 2 & 0 & a \\ 1 & a & b \end{array}\right].$$

Simple calculations yield

$$\mathcal{E}_C \tilde{\mathcal{U}} = \mathcal{A}_C \tilde{\mathcal{U}} \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \tilde{\mathcal{U}} = \begin{bmatrix} 0 & -a & a-2b \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}.$$

So $\lambda\mathcal{E}_C - \mathcal{A}_C$ has the only eigenvalue ∞. Clearly, $\mathbf{t}$ sends both the eigenvalues $-1, \infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ to $\infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$.

The relation about the spectral subspaces can be shown as follows. By (7) with $U := U_1$ and $T = \left[\begin{smallmatrix} -1 & -1 \\ 0 & -1 \end{smallmatrix}\right]$, we have

$$\hat{U} = \begin{bmatrix} \begin{bmatrix} a & b-a \end{bmatrix} T \\ \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{bmatrix} = \begin{bmatrix} -a & -b \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = V_1 \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This shows $\operatorname{span} \hat{U} = \mathcal{L}_{-1}^D (= \operatorname{span} V_1)$.

By (15) and the fact that $r_1 = 0$, we have

$$\tilde{U} = U_1 + \hat{U} = \begin{bmatrix} 0 & -a \\ 2 & 0 \\ 0 & 2 \end{bmatrix},$$

which is just the submatrix formed by the first and second columns of $\tilde{\mathcal{U}}$.

There is an interesting observation. Although $\infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$ is transformed from two different eigenvalues $-1, \infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$, it has a single 3×3 Jordan block.

Example 3 Consider the D-type pencil from Example 1,

$$\lambda \mathcal{E}_D - \mathcal{A}_D = \lambda \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline -1 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right] - \left[\begin{array}{c|cc} 0 & 1 & 1 \\ \hline -1 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right].$$

We had

$$\begin{aligned} &\mathcal{E}_D U_1 \cdot 0 = \mathcal{A}_D U_1, \quad 0 \cdot V_1^* \mathcal{E}_D = V_1^* \mathcal{A}_D; \\ &\mathcal{E}_D U_2 = \mathcal{A}_D U_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad V_2^* \mathcal{E}_D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^* V_2^* \mathcal{A}_D, \end{aligned}$$

where

$$U_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad V_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}; \qquad U_2 = \begin{bmatrix} 0 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Under the transformation $\mathbf{t}$, the corresponding C-type pencil is

$$\lambda \mathcal{E}_C - \mathcal{A}_C = \lambda \left[\begin{array}{c|cc} 0 & 2 & 1 \\ \hline -2 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right] - \left[\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right].$$

Let

$$\tilde{U}_1 = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}, \qquad \tilde{V}_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \qquad \tilde{U}_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}.$$

We have

$$\begin{aligned} -\mathcal{E}_C \tilde{U}_1 &= \mathcal{A}_C \tilde{U}_1, & -\tilde{V}_1^* \mathcal{E}_C &= \tilde{V}_1^* \mathcal{A}_C \\ \mathcal{E}_C \tilde{V}_1 &= \mathcal{A}_C \tilde{V}_1, & \tilde{U}_1^* \mathcal{E}_C &= \tilde{U}_1^* \mathcal{A}_C \\ \mathcal{E}_C \tilde{U}_2 &= \mathcal{A}_C \tilde{U}_2 \cdot 0, & \tilde{U}_2^* \mathcal{E}_C &= 0 \cdot \tilde{U}_2^* \mathcal{A}_C. \end{aligned}$$

Note that $\tilde{U}_1, \tilde{V}_1$ are just the matrices $\tilde{U}$, $\tilde{V}$ in (13) derived from $U := U_1$, $V := V_1$ and $T = 0$. It is clear that $-1 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$ is transformed from $0 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$, and both $1, \infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$ are transformed from $\infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$.

5 Conclusion

We have introduced a one-to-one transformation between the D-type and C-type matrix pencils that arise in optimal and robust control. We have given the precise descriptions about the equivalence eigenvalue and deflating subspace relations between the pencils under the transformation. The proposed transformation may be implemented to relate the discrete-time and continuous-time problems from each specific topic in optimal and robust control (such as the LQ control, H_2 control, H_∞ control, etc.), so that the two types of control problems can be studied, both theoretically and numerically, in a unified way. Every topic has its own aims, properties, and conditions, and the associated matrix pencils also have extra structures and properties. Therefore, the implementation is not trivial. In order to precisely relate the discrete-time and continuous-time control problems much work still needs to be carefully done.

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Appendix A

Proof of Theorem 10.

(a) The equation $\mathcal{E}_D U T = \mathcal{A}_D U$ yields

$$FU_2T = GU_2, \qquad -G^*U_1T = -F^*U_1 + DU_2.$$

Multiplying the first equation by -1 and post-multiplying the second equation by T, we have

$$-GU_2 = -FU_2T, \qquad F^*(U_1T) = (G^*(U_1T) + DU_2)T.$$

The last two equations can be expressed by a single equation

$$\begin{bmatrix} 0 & -G \\ F^* & 0 \end{bmatrix} \begin{bmatrix} U_1T \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 & -F \\ G^* & D \end{bmatrix} \begin{bmatrix} U_1T \\ U_2 \end{bmatrix} T,$$

which is just $\mathcal{E}_D^* \hat{U} = \mathcal{A}_D^* \hat{U} T$, or equivalently, $T^{-*}\hat{U}^*\mathcal{E}_D = \hat{U}^*\mathcal{A}_D$.

Conversely, since T is nonsingular, matrix $\hat{U}$ can be expressed as $\begin{bmatrix} U_1T \\ U_2 \end{bmatrix}$. Then by reversing the above procedure we obtain $\mathcal{E}_D U T = \mathcal{A}_D U$ from $T^{-*}\hat{U}^*\mathcal{E}_D = \hat{U}^*\mathcal{A}_D$.

Because the pencil $\lambda\mathcal{E}_D^* - \mathcal{A}_D^*$ has the same block structure as $\lambda\mathcal{E}_D - \mathcal{A}_D$, the equivalence relation between $S^*V^*\mathcal{E}_D = V^*\mathcal{A}_D$ and $\mathcal{E}_D\hat{V}S^{-1} = \mathcal{A}_D\hat{V}$ can be established in the same way.

We now prove that $\det V^*\mathcal{E}_D U \neq 0$ and $\det \hat{U}^*\mathcal{E}_D\hat{V} \neq 0$ are equivalent. By using the relations

$$FU_2T = GU_2, \qquad GV_2S = FV_2,$$

which are from $\mathcal{E}_D UT = \mathcal{A}_D U$ and $S^*V^*\mathcal{E}_D = V^*\mathcal{A}_D$, respectively, we have

$$\begin{aligned} \hat{U}^*\mathcal{E}_D\hat{V} &= T^*U_1^*FV_2 - U_2^*G^*V_1S = T^*U_1^*GV_2S - T^*U_2^*F^*V_1S \\ &= T^*(U_1^*GV_2 - U_2^*F^*V_1)S = -T^*(V^*\mathcal{E}_D U)^*S. \end{aligned}$$

Since $\det T, \det S \neq 0$, one has $\det \hat{U}^*\mathcal{E}_D\hat{V} \neq 0$ if and only if $\det V^*\mathcal{E}_D U \neq 0$.

The above relations show that if $\lambda_0 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ with algebraic multiplicity ℓ, then $\bar{\lambda}_0^{-1} \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ with algebraic multiplicity at least ℓ. The algebraic multiplicity of $\bar{\lambda}_0^{-1}$ can not be bigger, since otherwise by the duality, the algebraic multiplicity of λ_0 would be bigger than ℓ. Thus, both λ_0 and $\bar{\lambda}_0$ have the same algebraic multiplicity ℓ.

The rest follow simply from the above relations and arguments.

(b) The existence of U, V in (8) is shown in Lemma 9 with $\lambda_0 = 0$.

The equation $\mathcal{E}_D UT = \mathcal{A}_D U$ implies

$$\begin{aligned} &FU_{22}T_{22} = GU_{22}, \\ &G^*U_{11}T_{11} = F^*U_{11}, \\ &-G^*(U_{11}T_{12} + U_{12}T_{22}) = -F^*U_{12} + DU_{22}. \end{aligned} \tag{16}$$

By using the above equations, we have

$$\begin{aligned} \mathcal{A}_D^*\hat{U}\hat{T} &= \begin{bmatrix} 0 & -F \\ G^* & D \end{bmatrix} \begin{bmatrix} U_{11} & U_{11}T_{12} + U_{12}T_{22} \\ 0 & U_{22} \end{bmatrix} \begin{bmatrix} T_{11} & T_{11}T_{12} \\ 0 & T_{22} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -FU_{22}T_{22} \\ G^*U_{11}T_{11} & G^*U_{11}T_{11}T_{12} + [G^*(U_{11}T_{12} + U_{12}T_{22}) + DU_{22}]T_{22} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -GU_{22} \\ F^*U_{11} & F^*U_{11}T_{12} + F^*U_{12}T_{22} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -G \\ F^* & 0 \end{bmatrix} \begin{bmatrix} U_{11} & U_{11}T_{12} + U_{12}T_{22} \\ 0 & U_{22} \end{bmatrix} = \mathcal{E}_D^*\hat{U}, \end{aligned}$$

or equivalently, $\hat{U}^*\mathcal{E}_D = \hat{T}^*\hat{U}^*\mathcal{A}_D$.

Next, we show that $\operatorname{rank} \mathcal{E}_D U = \ell_0$ implies $\operatorname{rank} \hat{U}^*\mathcal{A}_D = \ell_0$. By the last equation in (16) we have

$$\mathcal{A}_D^*\hat{U} = \begin{bmatrix} 0 & -FU_{22} \\ G^*U_{11} & G^*(U_{11}T_{12} + U_{12}T_{22}) + DU_{22} \end{bmatrix} = \begin{bmatrix} 0 & -FU_{22} \\ G^*U_{11} & F^*U_{12} \end{bmatrix}.$$

If $\operatorname{rank} \mathcal{A}_D^* \hat{U} < \ell_0$, then there exists a nonzero vector $x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ with $x_1 \in \mathbb{C}^{r_1}$ and $x_2 \in \mathbb{C}^{\ell_0 - r_1}$, such that $\mathcal{A}_D^* \hat{U} x = 0$. This implies

$$-FU_{22}x_2 = 0, \qquad G^* U_{11} x_1 + F^* U_{12} x_2 = 0.$$

The vector x_2 is nonzero. Otherwise $x_1 \neq 0$ and $G^* U_{11} x_1 = 0$, but then

$$\mathcal{E}_D U \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & FU_{22} \\ -G^* U_{11} & -G^* U_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -G^* U_{11} x_1 \end{bmatrix} = 0,$$

contradicting to $\operatorname{rank} \mathcal{E}_D U = \ell_0$.

Now combining $G^* U_{11} x_1 + F^* U_{12} x_2 = 0$ with $G^* U_{11} T_{11} = F^* U_{11}$, the second equation in (16), we have

$$G^* \begin{bmatrix} U_{11} & U_{12} x_2 \end{bmatrix} \begin{bmatrix} T_{11} & -x_1 \\ 0 & 0 \end{bmatrix} = F^* \begin{bmatrix} U_{11} & U_{12} x_2 \end{bmatrix}.$$

Since U_{11} is a basis matrix of the spectral subspace $\lambda G^* - F^*$ corresponding to the eigenvalue 0, we have

$$U_{12} x_2 = U_{11} x_3$$

for some vector $x_3 \in \mathbb{C}^{r_1}$.

Now $\left[\begin{smallmatrix} -x_3 \\ x_2 \end{smallmatrix}\right] \neq 0$, but

$$\mathcal{E}_D U \begin{bmatrix} -x_3 \\ x_2 \end{bmatrix} = \begin{bmatrix} FU_{22} x_2 \\ G^* (U_{11} x_3 - U_{12} x_2) \end{bmatrix} = 0,$$

which again contradicts to $\operatorname{rank} \mathcal{E}_D U = \ell_0$. Therefore, $\operatorname{rank} \mathcal{A}_D^* \hat{U} = \ell_0$.

It becomes obvious that the algebraic multiplicity of $\infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ is at least ℓ_0, and the columns of $\hat{U}$ span an associated left deflating subspace.

By the duality, the result associated with $\hat{V}$ can be proved in the same way. □

Appendix B

Proof of Theorem 16. Let $(\tilde{\mathcal{E}}, \tilde{\mathcal{A}}) = \mathbf{c}(\mathcal{E}_D, \mathcal{A}_D) = (\mathcal{A}_D + \mathcal{E}_D, \mathcal{A}_D - \mathcal{E}_D)$.

(a) It is easily verified that the equations

$$\mathcal{E}_D U T = \mathcal{A}_D U, \quad S^* V^* \mathcal{E}_D = V^* \mathcal{A}_D$$

are equivalent to

$$\tilde{\mathcal{E}} U \mathbf{c}(T) = \tilde{\mathcal{A}} U, \quad \mathbf{c}(S^*) V^* \tilde{\mathcal{E}} = V^* \tilde{\mathcal{A}}.$$

By Proposition 14, $\Lambda(\mathbf{c}(T)) = \Lambda(\mathbf{c}(S^*)) = \{\mu_0\}$, where $\mu_0 = \mathbf{c}(\lambda_0)$. Because $\lambda_0 \neq -1, \infty$, we have $\mu_0 \neq \infty, 1$.

By using (11), the equation $\tilde{\mathcal{E}} U \mathbf{c}(T) = \tilde{\mathcal{A}} U$ yields

$$\tilde{F} U_2 \mathbf{c}(T) = \tilde{G} U_2, \quad -\tilde{F}^* U_1 \mathbf{c}(T) = \tilde{G}^* U_1 + \tilde{D} U_2 (I - \mathbf{c}(T)).$$

Multiplying the first equation by 2 and post-multiplying the second equation by $T + I$, by using the fact that $\mathbf{c}(T)$ and $T + I$ commute, and the identity

$$I - \mathbf{c}(T) = I - (T - I)(T + I)^{-1} = 2(T + I)^{-1},$$

we have

$$\tilde{F}(2U_2)\mathbf{c}(T) = \tilde{G}(2U_2), \quad -\tilde{F}^* U_1 (T + I)\mathbf{c}(T) = \tilde{G}^* U_1 (T + I) + \tilde{D}(2U_2),$$

or equivalently,

$$\begin{bmatrix} 0 & \tilde{F} \\ -\tilde{F}^* & 0 \end{bmatrix} \begin{bmatrix} U_1(I + T) \\ 2U_2 \end{bmatrix} \mathbf{c}(T) = \begin{bmatrix} 0 & \tilde{G} \\ \tilde{G}^* & \tilde{D} \end{bmatrix} \begin{bmatrix} U_1(I + T) \\ 2U_2 \end{bmatrix},$$

i.e., $\mathcal{E}_C \tilde{U} \tilde{T} = \mathcal{A}_C \tilde{U}$.

Conversely, suppose that $\tilde{U}$ satisfies $\mathcal{E}_C \tilde{U} \tilde{T} = \mathcal{A}_C \tilde{U}$, with $\Lambda(\tilde{T}) = \{\mu_0\}$ and $\mu_0 \neq \infty, 1$. Let $T = \mathbf{c}^{-1}(\tilde{T})$. Then $\Lambda(T) = \{\lambda_0\}$, where $\lambda_0 = \mathbf{c}^{-1}(\mu_0) \neq -1, \infty$. Since $I + T$ is invertible, we can express $\tilde{U} = \begin{bmatrix} U_1(I+T) \\ 2U_2 \end{bmatrix}$. Define $U := \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$. By reversing the above steps we can show that U and T satisfy $\mathcal{E}_D U T = \mathcal{A}_D U$.

Similarly, we can show that $S^* V^* \mathcal{E}_D = V^* \mathcal{A}_D$ and $\tilde{S}^* \tilde{V}^* \mathcal{E}_C = \tilde{V}^* \mathcal{A}_C$ are equivalent.

We now prove that $\det V^* \mathcal{E}_D U \neq 0$ and $\det \tilde{V}^* \mathcal{E}_C \tilde{U} \neq 0$ are equivalent. From the equations $\mathcal{E}_D U T = \mathcal{A}_D U$ and $S^* V^* \mathcal{E}_D = V^* \mathcal{A}_D$, we have

$$F U_2 T = G U_2, \qquad S^* V_2^* G^* = V_2^* F^*.$$

Then

$$\begin{aligned}
\tilde{V}^* \mathcal{E}_C \tilde{U} &= 2(I + S)^* V_1^* (G + F) U_2 - 2 V_2^* (G + F)^* U_1 (I + T) \\
&= 2(I + S)^* V_1^* (G U_2 + F U_2) - 2(V_2^* G^* + V_2^* F^*) U_1 (I + T) \\
&= 2(I + S)^* V_1^* (F U_2 T + F U_2) - 2(V_2^* G^* + S^* V_2^* G^*) U_1 (I + T) \\
&= 2(I + S)^* V_1^* F U_2 (I + T) - 2(I + S)^* V_2^* G^* U_1 (I + T) \\
&= 2(I + S)^* (V_1^* F U_2 - V_2^* G^* U_1)(I + T) \\
&= 2(I + S)^* (V^* \mathcal{E}_D U)(I + T).
\end{aligned}$$

Since $(I+S)^*$ and $(I+T)$ are nonsingular, one has $\det \tilde{V}^* \mathcal{E}_C \tilde{U} \neq 0$ if and only if $\det V^* \mathcal{E}_D U \neq 0$. The rest are obvious.

(b) The existence of U in (14) is shown in Lemma 9 with $\lambda_0 = -1$. The equation $\mathcal{E}_D U T = \mathcal{A}_D U$ implies $\tilde{\mathcal{E}} U = \tilde{\mathcal{A}} U \hat{T}$, where

$$\begin{aligned} \hat{T} &= (T+I)(T-I)^{-1} \\ &= \begin{bmatrix} (T_{11}+I)(T_{11}-I)^{-1} & -2(T_{11}-I)^{-1}T_{12}(T_{22}-I)^{-1} \\ 0 & (T_{22}+I)(T_{22}-I)^{-1} \end{bmatrix} \\ &=: \begin{bmatrix} \hat{T}_{11} & \hat{T}_{12} \\ 0 & \hat{T}_{22} \end{bmatrix}. \end{aligned}$$

By (11), the equation $\tilde{\mathcal{E}} U = \tilde{\mathcal{A}} U \hat{T}$ then yields

$$\begin{aligned} &\tilde{F} U_{22} = \tilde{G} U_{22} \hat{T}_{22}, \\ &-\tilde{F}^* U_{11} = \tilde{G}^* U_{11} \hat{T}_{11}, \\ &-\tilde{F}^* U_{12} = \tilde{G}^* (U_{11} \hat{T}_{12} + U_{12} \hat{T}_{22}) + \tilde{D} U_{22} (\hat{T}_{22} - I). \end{aligned}$$

Multiplying the first and the second equations by 2, post-multiplying the 3rd equation by $(T_{22}+I)$, and using $\hat{T}_{22} - I = 2(T_{22}-I)^{-1}$, we have

$$\begin{aligned} &\tilde{F}(2U_{22}) = \tilde{G}(2U_{22})\hat{T}_{22}, \\ &-\tilde{F}^*(2U_{11}) = \tilde{G}^*(2U_{11})\hat{T}_{11}, \\ &-\tilde{F}^* U_{12}(T_{22}+I) = \tilde{G}^*[-(2U_{11})(T_{11}-I)^{-1}T_{12}\hat{T}_{22} + U_{12}(T_{22}+I)\hat{T}_{22}] \\ &\qquad\qquad + \tilde{D}(2U_{22})\hat{T}_{22}. \end{aligned}$$

These equations yield $\mathcal{E}_C \tilde{U} = \mathcal{A}_C \tilde{U} \tilde{T}$.

We now show that $\operatorname{rank} \mathcal{E}_D U = \ell_{-1}$ implies $\operatorname{rank} \mathcal{A}_C \tilde{U} = \ell_{-1}$. From the equation $\mathcal{E}_D U T = \mathcal{A}_D U$, we have

$$\begin{aligned} &F U_{22} T_{22} = G U_{22}, \\ &G^* U_{11} T_{11} = F^* U_{11}, \\ &-G^*(U_{11}T_{12} + U_{12}T_{22}) = -F^* U_{12} + D U_{22}. \end{aligned}$$

By using these equations,

$$\begin{aligned} \mathcal{A}_C \tilde{U} &= \begin{bmatrix} 0 & 2(G-F)U_{22} \\ 2(G-F)^* U_{11} & (G-F)^* U_{12}(T_{22}+I) + 2DU_{22} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2FU_{22}(T_{22}-I) \\ 2G^* U_{11}(I-T_{11}) & (G+F)^* U_{12}(I-T_{22}) - 2G^* U_{11} T_{12} \end{bmatrix}. \end{aligned}$$

If $\operatorname{rank} \mathcal{A}_C\tilde{U} < \ell_{-1}$, then there is a nonzero vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $\mathcal{A}_C\tilde{U}x = 0$. This implies

$$FU_{22}(T_{22} - I)x_2 = 0 \tag{17}$$

and

$$2G^*U_{11}(I - T_{11})x_1 + ((G + F)^*U_{12}(I - T_{22}) - 2G^*U_{11}T_{12})\, x_2 = 0. \tag{18}$$

We have $x_2 \neq 0$. Otherwise $x_1 \neq 0$ and $G^*U_{11}(I - T_{11})x_1 = 0$. Because $\Lambda(T_{11}) = \{-1\}$, one has $(I - T_{11})x_1 \neq 0$. Then

$$\mathcal{E}_D U \begin{bmatrix} (I - T_{11})x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -G^*U_{11}(I - T_{11})x_1 \end{bmatrix} = 0,$$

contradicting to $\operatorname{rank} \mathcal{E}_D U = \ell_{-1}$. Now rewrite (18) as

$$G^*[2U_{11}((I - T_{11})x_1 - T_{12}x_2) + U_{12}(I - T_{22})x_2] = -F^*U_{12}(I - T_{22})x_2.$$

Combining it with $G^*U_{11}T_{11} = F^*U_{11}$, which is from $\mathcal{E}_D UT = \mathcal{A}_D U$, we have

$$\begin{aligned} &G^* \begin{bmatrix} U_{11} & -U_{12}(I - T_{22})x_2 \end{bmatrix} \begin{bmatrix} T_{11} & 2((I - T_{11})x_1 - T_{12}x_2) \\ 0 & -1 \end{bmatrix} \\ &= F^* \begin{bmatrix} U_{11} & -U_{12}(I - T_{22})x_2 \end{bmatrix}. \end{aligned}$$

Since U_{11} is a basis matrix of the spectral subspace $\lambda G^* - F^*$ corresponding to the eigenvalue -1, we have

$$U_{12}(I - T_{22})x_2 = U_{11}x_3 \tag{19}$$

for some vector x_3. Let $z = \begin{bmatrix} -x_3 \\ (I - T_{22})x_2 \end{bmatrix}$. Because $\Lambda(T_{22}) = \{-1\}$ and $x_2 \neq 0$, we have $z \neq 0$. Then (17) and (19) imply

$$\mathcal{E}_D U z = \begin{bmatrix} FU_{22}(I - T_{22})x_2 \\ -G^*(-U_{11}x_3 + U_{12}(I - T_{22})x_2) \end{bmatrix} = 0,$$

which again contradicts to $\operatorname{rank} \mathcal{E}_D U = \ell_{-1}$. Therefore, $\operatorname{rank} \mathcal{A}_C\tilde{U} = \ell_{-1}$.

The rest are obvious.

(c) By Theorem 10, $\ell_\infty \geq \ell_0$. From part (a), $0 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ and $-1 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$ have the same algebraic multiplicity, i.e., $\ell_0 = \tilde{\ell}_{-1}$. By Proposition 8 (a), $-1, 1 \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$ are paired and have the same algebraic multiplicity. So we have

$$\tilde{\ell}_1 = \tilde{\ell}_{-1} = \ell_0. \tag{20}$$

Under the transformation $\mathbf{t}$, the eigenvalue $-1 \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ is transformed to $\infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$. From parts (a) and (b), the eigenvalue $\infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ has to be transformed to $1, \infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$. From the relation between $-1, \infty \in \Lambda(\mathcal{E}_D, \mathcal{A}_D)$ and $1, \infty \in \Lambda(\mathcal{E}_C, \mathcal{A}_C)$ discussed above, we have

$$\ell_{-1} + \ell_{\infty} = \tilde{\ell}_1 + \tilde{\ell}_{\infty}. \tag{21}$$

The result then follows from (20) and (21), and the arguments given above. □

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