Transformations Between Discrete-time and Continuous-time Algebraic Riccati Equations

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Abstract

We introduce a transformation between the discrete-time and continuous-time algebraic Riccati equations. We show that under mild conditions the two algebraic Riccati equations can be transformed from one to another, and both algebraic Riccati equations share common Hermitian solutions. The transformation also sets up the relations about the properties, commonly in system and control setting, that are imposed in parallel to the coefficient matrices and Hermitian solutions of two algebraic Riccati equations. The transformation is simple and all the relations can be easily derived. We also introduce a generalized transformation that requires weaker conditions. The proposed transformations may provide a unified tool to develop the theories and numerical methods for the algebraic Riccati equations and the associated system and control problems.

Keywords Algebraic Riccati equation, reducing subspace, eigenvalue, controllability, stability, regularizability, Cayley transformation.
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1 Introduction

We consider the relation between two types of algebraic Riccati equations (AREs). The first type is the discrete-time algebraic Riccati equation (DARE)

$$\begin{aligned}\mathbf{R}_d(X) \quad := \quad & A_d^* X A_d - E_d^* X E_d + M_d \\ & -(A_d^* X B_d + N_d)(R_d + B_d^* X B_d)^{-1}(A_d^* X B_d + N_d)^* = 0, \quad (1)\end{aligned}$$

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where $E_d, A_d, M_d \in \mathbb{C}^{n,n}$, $R_d \in \mathbb{C}^{p,p}$, $B_d, N_d \in \mathbb{C}^{n,p}$, and M_d, R_d are Hermitian. The second type is the continuous-time algebraic Riccati equation (CARE)

$$\begin{aligned}\mathbf{R}_c(X) \;:=\; & E_c^* X A_c + A_c^* X E_c + M_c \\ & -(E_c^* X B_c + N_c) R_c^{-1} (E_c^* X B_c + N_c)^* = 0, \end{aligned} \tag{2}$$

where $E_c, A_c, M_c \in \mathbb{C}^{n,n}$, $R_c \in \mathbb{C}^{p,p}$, $B_c, N_c \in \mathbb{C}^{n,p}$, and M_c, R_c are Hermitian.

The AREs play a fundamental role in linear optimal and robust control. For instance, the solvability of the discrete-time linear quadratic optimal control problem,

$$\begin{aligned} & \min_{u_k} \frac{1}{2} \sum_{k=0}^{\infty} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^* \begin{bmatrix} M_d & N_d \\ N_d^* & R_d \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \\ & \text{subject to} \quad E_d x_{k+1} = A_d x_k + B_d u_k \qquad x_0 = x^0 \end{aligned}$$

with $M_d^* = M_d$, $R_d^* = R_d$, depends on the solvability of the DARE (1), e.g., [21, 24, 2, 3, 19, 16, 12]. Likewise, the solvability of the continuous-time linear quadratic optimal problem,

$$\begin{aligned} & \min_{u} \frac{1}{2} \int_0^{\infty} \begin{bmatrix} x \\ u \end{bmatrix}^* \begin{bmatrix} M_c & N_c \\ N_c^* & R_c \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt \\ & \text{subject to} \quad E_c \dot{x} = A_c x + B_c u \qquad x(0) = x^0 \end{aligned}$$

with $M_c^* = M_c$, $R_c^* = R_c$, depends on the solvability of the CARE (2), e.g., [13, 1, 15, 17, 24, 19, 16]. Due to the important applications in system and control, in the past decades the AREs have been extensively studied. The theoretical results and numerical methods have been well developed, see, e.g., [13, 1, 15, 17, 21, 24, 5, 6, 18, 19, 16, 4, 8, 12, 7] and the references therein. Although in literature the AREs (1) and (2) are usually treated separately, it is well-known that, due to the similar background, their structures and properties appear in parallel. For instance, the AREs (1) and (2) are characterized by the same type of coefficient matrix tuples $(E_d, A_d, B_d, M_d, N_d, R_d)$ and $(E_c, A_c, B_c, M_c, N_c, R_c)$. Both Riccati operators $\mathbf{R}_c$ and $\mathbf{R}_d$ can be considered as transformations in the set of Hermitian matrices. For the AREs arising from system and control, special concepts and properties, such as controllability, stability, etc., are usually introduced in parallel to the coefficient matrices. These similarities lead to the investigation on equivalence relations between the two AREs. In [20], it is shown that under certain conditions the AREs can be related by the Cayley transformation. However, the conditions may be too strong for many AREs and the relation between two sets of coefficient matrices may be very complicated. The transformation proposed in [11] is less restrictive. However, it still needs certain invertibility conditions.

In this paper we introduce the following invertible transformation for the AREs (1) and (2). Given $(E_d, A_d, B_d, M_d, N_d, R_d)$, let W be an invertible matrix such that $\begin{bmatrix} A_d + E_d & B_d \end{bmatrix} W^{-1} = \begin{bmatrix} H & 0 \end{bmatrix}$ with $H \in \mathbb{C}^{n,n}$. Then we define

$$\mathbf{f}_W : (E_d, A_d, B_d, M_d, N_d, R_d) \mapsto (E_c, A_c, B_c, M_c, N_c, R_c),$$

$$\begin{aligned} \begin{bmatrix} E_c & 0 \\ A_c & B_c \end{bmatrix} &= \frac{\sqrt{2}}{2} \begin{bmatrix} A_d + E_d & B_d \\ A_d - E_d & B_d \end{bmatrix} W^{-1}, \\ \begin{bmatrix} M_c & N_c \\ N_c^* & R_c \end{bmatrix} &= W^{-*} \begin{bmatrix} M_d & N_d \\ N_d^* & R_d \end{bmatrix} W^{-1}. \end{aligned}$$

We will show that under mild conditions the AREs related by the transformation $\mathbf{f}_W$ share the same Hermitian solutions.

The Hermitian solutions of the AREs are related to reducing (or deflating) subspaces of certain matrix pairs determined by the coefficient matrices of the AREs ([17, 19, 16]). In [26], an equivalence transformation between the corresponding matrix pairs and reducing subspaces was given. We will see below that the transformation in [26] can be considered as an intermediate one of $\mathbf{f}_W$. However, the relation between ARE solutions and reducing subspaces is not an equivalence relation ([16]). For this reason, we will directly study the relation between the solutions of two types of AREs under the transformation $\mathbf{f}_W$.

The paper is organized as follows. Section 2 gives some necessary definitions and properties about matrix pairs and AREs, and some other auxiliary results. The proposed transformation has a strong tie with the Cayley transformation. So a brief review about the Cayley transformation is also given in this section. Section 3 formally introduces the above transformation. Relations about some controllability properties of the coefficient matrices are also described. The transformation introduced in [26] is also presented in this section. Section 4 gives several sufficient conditions under which a DARE can be transformed to a CARE by the proposed transformation. It also gives the relation between their Hermitian solutions. Section 5 is parallel to Section 4.

It gives sufficient conditions under which a CARE can be transformed to a DARE by the inverse transformation. Section 6 introduces a generalized transformation. Section 7 contains the conclusions.

Throughout the paper, $\mathbb{R}$ denotes the set of real numbers. $\mathbb{C}, \mathbb{C}^k, \mathbb{C}^{m,q}$ denote the set of complex numbers, the k-dimensional complex vector space, and the space of $m \times q$ complex matrices, respectively. $\mathbb{H}^{k,k}$ denotes the set of $k \times k$ complex Hermitian matrices. $\mathbb{C}^+$, $\mathbb{C}^-$, and $\mathbb{C}^0$ denote the sets of complex numbers with positive , negative, and zero real parts, respectively. $\mathbb{O}^-$, $\mathbb{O}^+$, and $\mathbb{O}^0$ denote the sets of complex numbers inside, outside, and on the unit circle, respectively. $\operatorname{rank} X$ is the rank of matrix X. $\operatorname{null} X$ is the null space of matrix X. $\operatorname{span} X$ is the subspace spanned by the columns of matrix X. X^* is the complex conjugate transpose of X. $X^{-*} = (X^*)^{-1}$. $0_{p\times q}$ (0_p) is the $p \times q$ $(p \times p)$ zero matrix and I_p is the $p \times p$ identity matrix. When the sizes are obvious from the context, they are simply denoted by 0 and I, respectively.

2 Preliminaries

Definition 1 *Two matrix pairs* $(\mathcal{E}, \mathcal{A}), (\mathcal{F}, \mathcal{B}) \in \mathbb{C}^{m,q} \times \mathbb{C}^{m,q}$ *are called* equivalent *if there exist nonsingular matrices* $\mathcal{X}$ *and* $\mathcal{Y}$ *such that*

$$(\mathcal{F}, \mathcal{B}) = (\mathcal{X}\mathcal{E}\mathcal{Y},\ \mathcal{X}\mathcal{A}\mathcal{Y}).$$

Definition 2 *Consider the matrix pair* $(\mathcal{E}, \mathcal{A}) \in \mathbb{C}^{m,q} \times \mathbb{C}^{m,q}$. *If* $m = q$ *and* $\det(\mathcal{A} - \lambda\mathcal{E}) \neq 0$ *for some* $\lambda \in \mathbb{C}$, *the pair* $(\mathcal{E}, \mathcal{A})$ *is* regular. *If either* $m \neq q$ *or* $m = q$ *and* $\det(\mathcal{A} - \lambda\mathcal{E}) = 0$ *for all* $\lambda \in \mathbb{C}$, *the pair* $(\mathcal{E}, \mathcal{A})$ *is* singular.

Theorem 3 ([14, 9]). *Any pair* $(\mathcal{E}, \mathcal{A}) \in \mathbb{C}^{m,q} \times \mathbb{C}^{m,q}$ *is equivalent to a pair of the block form*

$$(\operatorname{diag}(0,\ E_r,\ E_l,\ E_g),\ \operatorname{diag}(0,\ A_r,\ A_l,\ A_g)), \tag{3}$$

where $\alpha E_r - \beta A_r$ *and* $\alpha E_l - \beta A_l$ *have full row and column ranks, respectively, for all* $\alpha, \beta \in \mathbb{C}$ *not both zero, and* (E_g, A_g) *is regular. The regular subpair* (E_g, A_g) *is unique (up to equivalence transformations).*

When $(\mathcal{E}, \mathcal{A})$ is regular, any number $\lambda_0 \in \mathbb{C}$ satisfying $\det(\lambda_0\mathcal{E} - \mathcal{A}) = 0$ is a *finite eigenvalue* of $(\mathcal{E}, \mathcal{A})$. If $\mathcal{E}$ is singular, then ∞ is also an eigenvalue of

$(\mathcal{E}, \mathcal{A})$. For a general pair $(\mathcal{E}, \mathcal{A})$, its eigenvalues are just those of the sub-pair (E_g, A_g) defined in (3). We denote by $\Lambda(\mathcal{E}, \mathcal{A})$ the set of all (finite and infinite) eigenvalues of $(\mathcal{E}, \mathcal{A})$.

Definition 4 *For a given subspace $\mathcal{S}$, X is called a* basis matrix *of $\mathcal{S}$ if the matrix X has full column rank and* $\operatorname{span} X = \mathcal{S}$.

Definition 5 *Consider the pair $(\mathcal{E}, \mathcal{A}) \in \mathbb{C}^{m,q} \times \mathbb{C}^{m,q}$.*

1. *If $U \in \mathbb{C}^{q,k}$ has full column rank and satisfies*

$$\mathcal{E}U = YS, \qquad \mathcal{A}U = YT,$$

where $Y \in \mathbb{C}^{m,k}$ has full column rank and $S, T \in \mathbb{C}^{k,k}$, then U is a basis matrix of a right reducing subspace *of $(\mathcal{E}, \mathcal{A})$ associated with the sub-pair (S, T).*

2. *V is a basis matrix of a* left reducing subspace *of $(\mathcal{E}, \mathcal{A})$ associated with (S, T) if it is a basis matrix of a right reducing subspace of $(\mathcal{E}^*, \mathcal{A}^*)$ associated with (S^*, T^*).*

Definition 6 *Consider $(\mathcal{E}, \mathcal{A}) \in \mathbb{C}^{m,m} \times \mathbb{C}^{m,m}$.*

1. *The pair $(\mathcal{E}, \mathcal{A})$ is* C-stable *(resp.* C-semi-stable*), if $\Lambda(\mathcal{E}, \mathcal{A}) \subset \mathbb{C}^-$ (resp. $\Lambda(\mathcal{E}, \mathcal{A}) \subset \mathbb{C}^- \cup \mathbb{C}^0 \cup \{\infty\}$ and the eigenvalues in $\mathbb{C}^0 \cup \{\infty\}$ are semi-simple).*

2. *The pair $(\mathcal{E}, \mathcal{A})$ is* D-stable *(resp.* D-semi-stable*), if $\Lambda(\mathcal{E}, \mathcal{A}) \subset \mathbb{O}^-$ (resp. $\Lambda(\mathcal{E}, \mathcal{A}) \subset \mathbb{O}^- \cup \mathbb{O}^0$ and the eigenvalues in $\mathbb{O}^0$ are semi-simple).*

Definition 7 *Consider the matrix triplet $(E, A, B) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n} \times \mathbb{C}^{n,p}$.*

1. *The triplet is* controllable at $\lambda_0 \in \mathbb{C}$ *if* $\operatorname{rank}[A - \lambda_0 E,\ B] = n$.

2. *Given a set $\Omega \subseteq \mathbb{C}$, the triplet is* controllable in Ω *if it is controllable at every number in Ω.*

3. *The triplet is* controllable at infinity *if* $\operatorname{rank}[E,\ AT_\infty,\ B] = n$, *where T_∞ is a basis matrix of* $\operatorname{null} E$.

4. *The triplet is* controllable *if it is controllable in $\Omega = \mathbb{C}$.*

5. *The triplet is* C-stabilizable *(resp.,* D-stabilizable*) if it is controllable in* $\Omega = \mathbb{C}^+ \cup \mathbb{C}^0$ *(resp.,* $\Omega = \mathbb{O}^+ \cup \mathbb{O}^0$*).*

6. *The triplet is* regularizable *if it is controllable at some* $\lambda \in \mathbb{C}$.

Proposition 8 *Suppose* $(E, A, B) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,n} \times \mathbb{C}^{n,p}$.

(i) *There exist unitary matrices* P, Q *such that*

$$E = P \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix} Q,\; A = P \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} Q,\;\; B = P \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad (4)$$

where (E_{11}, A_{11}, B_1) *is controllable.*

(E, A, B) *is controllable if and only if* (E_{22}, A_{22}) *is void.*

(ii) *For any set* $\Omega \subseteq \mathbb{C}$*, there exist unitary matrices* P, Q *such that the triplet* (E, A, B) *has the form (4), where* (E_{11}, A_{11}, B_1) *is controllable in* Ω*, and* $\Lambda(E_{22}, A_{22}) \subseteq \Omega$.

(E, A, B) *is controllable in* Ω *if and only if* (E_{22}, A_{22}) *is void.*

Proof. **(i)** The factorization (4) is from [23].

(ii) It follows from (i) by reducing (E_{22}, A_{22}) further to a generalized Schur form and extracting the regular sub-pair with no eigenvalue in Ω to (E_{11}, A_{11}). □

Lemma 9 *Consider the Hermitian matrix*

$$A = \begin{array}{c} \\ n \\ p \end{array} \begin{array}{c} \begin{array}{cc} n & p \end{array} \\ \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix} \end{array} \in \mathbb{C}^{n+p,n+p},$$

Suppose that $\operatorname{rank} A = p$*, and* $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \in \mathbb{C}^{n+p,n}$ *(with* $Y_1 \in \mathbb{C}^{n,n}$*) is a basis matrix of* null A*. Then* $\det Y_1 \neq 0$ *if and only if* $\det A_{22} \neq 0$.

Proof. Necessity. Let $Y = \begin{bmatrix} Y_1 & 0 \\ Y_2 & I_p \end{bmatrix}$. Since $\det Y_1 \neq 0$, Y is nonsingular. Then from

$$Y^* A Y = \begin{bmatrix} 0 & 0 \\ 0 & A_{22} \end{bmatrix},$$

one has

$$p = \operatorname{rank} A = \operatorname{rank} Y^* A Y = \operatorname{rank} A_{22},$$

i.e., $\det A_{22} \neq 0$.

Sufficiency. Since $\operatorname{rank} A_{22} = \operatorname{rank} A = p$, $\begin{bmatrix} I_n \\ -A_{22}^{-1} A_{12}^* \end{bmatrix}$ is a basis matrix of $\operatorname{null} A$. Then $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} I_n \\ -A_{22}^{-1} A_{12}^* \end{bmatrix} T$ for some $T \in \mathbb{C}^{n,n}$ with $\det T \neq 0$. So $Y_1 = T$ is nonsingular. □

Define the dissipation operators

$$\begin{aligned}
\mathbf{D}_d(X) &= \begin{bmatrix} M_d & N_d \\ N_d^* & R_d \end{bmatrix} + \begin{bmatrix} E_d & 0 \\ A_d & B_d \end{bmatrix}^* \begin{bmatrix} -X & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} E_d & 0 \\ A_d & B_d \end{bmatrix} \\
&= \begin{bmatrix} A_d^* X A_d - E_d^* X E_d + M_d & A_d^* X B_d + N_d \\ B_d^* X A_d + N_d^* & B_d^* X B_d + R_d \end{bmatrix}, \\
\mathbf{D}_c(X) &= \begin{bmatrix} M_c & N_c \\ N_c^* & R_c \end{bmatrix} + \begin{bmatrix} E_c & 0 \\ A_c & B_c \end{bmatrix}^* \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \begin{bmatrix} E_c & 0 \\ A_c & B_c \end{bmatrix} \\
&= \begin{bmatrix} E_c^* X A_c + A_c^* X E_c + M_c & E_c^* X B_c + N_c \\ B_c^* X E_c + N_c^* & R_c \end{bmatrix},
\end{aligned}$$

and the matrix pairs

$$(\mathcal{E}_d, \mathcal{A}_d) = \left(\begin{bmatrix} 0 & E_d & 0 \\ -A_d^* & 0 & 0 \\ -B_d^* & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_d & B_d \\ -E_d^* & M_d & N_d \\ 0 & N_d^* & R_d \end{bmatrix} \right), \tag{5}$$

$$(\mathcal{E}_c, \mathcal{A}_c) = \left(\begin{bmatrix} 0 & E_c & 0 \\ -E_c^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_c & B_c \\ A_c^* & M_c & N_c \\ B_c^* & N_c^* & R_c \end{bmatrix} \right), \tag{6}$$

where $\mathbf{D}_d(X)$ and $(\mathcal{E}_d, \mathcal{A}_d)$ are related to the DARE (1), and $\mathbf{D}_c(X)$ and $(\mathcal{E}_c, \mathcal{A}_c)$ are related to the CARE (2). The following equivalence conditions can be verified directly. Most of the results can be found in [19, 16, 12].

Proposition 10 *Suppose $X \in \mathbb{H}^{n,n}$. Then the following statements are equivalent.*

(i) *X solves the DARE (1).*

(ii) $\operatorname{rank} \mathbf{D}_d(X) = \operatorname{rank}(R_d + B_d^* X B_d) = p$.

(iii) $\operatorname{rank} \mathbf{D}_d(X) = p$ *and* $\mathbf{D}_d(X) \begin{bmatrix} I_n \\ K_d \end{bmatrix} = 0$ *for some* $K_d \in \mathbb{C}^{p,n}$.

(iv) $\det(R_d + B_d^* X B_d) \neq 0$ *and there is a matrix* $K_d \in \mathbb{C}^{p,n}$ *such that the nonsingular matrices*

$$\mathcal{U}_d = \begin{bmatrix} I_n & XE_d & 0 \\ 0 & I_n & 0 \\ 0 & K_d & I_p \end{bmatrix}, \ \mathcal{Y}_d = \begin{bmatrix} I_n & X(A_d + B_d K_d) & B_d X \\ 0 & I_n & 0 \\ 0 & K_d & I_p \end{bmatrix}$$

satisfy

$$\begin{aligned} \mathcal{Y}_d^* \mathcal{E}_d \mathcal{U}_d &= \begin{bmatrix} 0 & E_d & 0 \\ -(A_d + B_d K_d)^* & 0 & 0 \\ -B_d^* & 0 & 0 \end{bmatrix}, \\ \mathcal{Y}_d^* \mathcal{A}_d \mathcal{U}_d &= \begin{bmatrix} 0 & A_d + B_d K_d & B_d \\ -E_d^* & 0 & 0 \\ 0 & 0 & R_d + B_d^* X B_d \end{bmatrix}. \end{aligned} \tag{7}$$

(v) $\det(R_d + B_d^* X B_d) \neq 0$ *and the matrices*

$$U_d = \mathcal{U}_d \begin{bmatrix} 0 \\ I_n \\ 0 \end{bmatrix} = \begin{bmatrix} XE_d \\ I_n \\ K_d \end{bmatrix}, \ Y_d = \mathcal{Y}_d^{-*} \begin{bmatrix} I_n \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} I_n \\ -A_d^* X \\ -B_d^* X \end{bmatrix} \tag{8}$$

satisfy

$$\mathcal{E}_d U_d = Y_d E_d, \qquad \mathcal{A}_d U_d = Y_d (A_d + B_d K_d),$$

i.e., U_d *is a basis matrix of a right reducing subspace of* $(\mathcal{E}_d, \mathcal{A}_d)$ *associated with* $(E_d, A_d + B_d K_d)$*.*

(vi) $\det(R_d + B_d^* X B_d) \neq 0$ *and the matrices*

$$V_d = \mathcal{Y}_d \begin{bmatrix} 0 \\ I_n \\ 0 \end{bmatrix} = \begin{bmatrix} X(A_d + B_d K_d) \\ I_n \\ K_d \end{bmatrix}, \ Z_d = \mathcal{U}_d^{-*} \begin{bmatrix} I_n \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} I_n \\ -E_d^* X \\ 0 \end{bmatrix} \tag{9}$$

satisfy

$$V_d^* \mathcal{E}_d = -(A_d + B_d K_d)^* Z_d^*, \qquad V_d^* \mathcal{A}_d = -E_d^* Z_d^*,$$

i.e., V_d *is a basis matrix of a left reducing subspace of* $(\mathcal{E}_d, \mathcal{A}_d)$ *associated with* $((A_d + B_d K_d)^*, E_d^*)$*.*

Moreover, if $X \in \mathbb{H}^{n,n}$ solves the DARE (1), the matrix K_d in (iii) – (vi) is the same. It depends on X and has the expression

$$K_d = -(R_d + B_d^* X B_d)^{-1}(B_d^* X A_d + N_d^*). \tag{10}$$

Proof. **(i)** $\Leftrightarrow$ **(ii)** easily follows from taking the Schur complement of $\mathbf{D}_d(X)$. **(ii)** $\Leftrightarrow$ **(iii)** follows from Lemma 9. **(i)** $\Leftrightarrow$ **(v)**, **(i)** $\Leftrightarrow$ **(vi)**, and **(iv)** $\Leftrightarrow$ **(v)** are straightforward. □

By eliminating B_d with pivoting $(R_d + B_d^* X B_d)$, the decomposition (7) can be reduced further to the block triangular form

$$\left(\begin{bmatrix} E_d & B_d(R_d + B_d^* X B_d)^{-1} B_d^* & 0 \\ 0 & -(A_d + B_d K_d)^* & 0 \\ 0 & B_d^* & 0 \end{bmatrix}, \begin{bmatrix} A_d + B_d K_d & 0 & 0 \\ 0 & -E_d^* & 0 \\ 0 & 0 & R_d + B_d^* X B_d \end{bmatrix} \right).$$

So we have

$$\Lambda(\mathcal{E}_d, \mathcal{A}_d) = \Lambda(E_d, A_d + B_d K_d) \cup \Lambda((A_d + B_d K_d)^*, E_d^*) \cup \Lambda(0_p, I_p). \tag{11}$$

Obviously, $\lambda \in \Lambda(E_d, A_d + B_d K_d)$ if and only if $\bar{\lambda}^{-1} \in \Lambda((A_d + B_d K_d)^*, E_d^*)$. (Here we assume that $0^{-1} = \infty$.) So the eigenvalues of $(\mathcal{E}_d, \mathcal{A}_d)$ appear in pairs $(\lambda, \bar{\lambda}^{-1})$, i.e., the spectrum has the *symplectic structure*. In fact, for any matrix pair of the form as $(\mathcal{E}_d, \mathcal{A}_d)$, its spectrum always has the symplectic structure, e.g., [19, 26]. However, when the DARE has an Hermitian solution, there are some extra properties about the eigenvalues on the unit circle. In this case, with an arbitrary Hermitian solution X and its corresponding K_d, we have (11). If $\lambda_0 \in \Lambda(\mathcal{E}_d, \mathcal{A}_d) \cap \mathbb{O}^0$, then $\bar{\lambda}_0^{-1} = \lambda_0$. So λ_0 must be contained in both $\Lambda(E_d, A_d + B_d K_d)$ and $\Lambda((A_d + B_d K_d)^*, E_d^*)$. The algebraic multiplicity of λ_0 in both spectra is obviously the same. Back to the original matrix pair $(\mathcal{E}_d, \mathcal{A}_d)$, the algebraic multiplicity of λ_0 must be even. It is also easily seen from (11) that for any Hermitian solution X and its corresponding K_d, $(E_d, A_d + B_d K_d)$ is regular if and only if $(\mathcal{E}_d, \mathcal{A}_d)$ is regular.

Similarly, for Hermitian solutions of the CARE (2) we have the following equivalence relations. Again, most of the results can be found in [19, 16, 12].

Proposition 11 *Suppose $X \in \mathbb{H}^{n,n}$. Then the following statements are equivalent.*

(i) *X solves the CARE (2).*

(ii) $\operatorname{rank} \mathbf{D}_c(X) = \operatorname{rank} R_c = p$.

(iii) $\operatorname{rank}\mathbf{D}_c(X) = p$ *and* $\mathbf{D}_c \left[\begin{smallmatrix} I_n \\ K_c \end{smallmatrix}\right] = 0$ *for some* $K_c \in \mathbb{C}^{p,n}$.

(iv) $\det R_c \neq 0$ *and there is a matrix* $K_c \in \mathbb{C}^{p,n}$ *such that the nonsingular matrix*

$$\mathcal{U}_c = \begin{bmatrix} I_n & XE_c & 0 \\ 0 & I_n & 0 \\ 0 & K_c & I_p \end{bmatrix}$$

satisfies

$$\begin{aligned} \mathcal{U}_c^* \mathcal{E}_c \mathcal{U}_c &= \begin{bmatrix} 0 & E_c & 0 \\ -E_c^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \mathcal{U}_c^* \mathcal{A}_c \mathcal{U}_c &= \begin{bmatrix} 0 & A_c + B_c K_c & B_c \\ (A_c + B_c K_c)^* & 0 & 0 \\ B_c^* & 0 & R_c \end{bmatrix}. \end{aligned} \tag{12}$$

(v) $\det R_c \neq 0$ *and the matrices*

$$U_c = \mathcal{U}_c \begin{bmatrix} 0 \\ I_n \\ 0 \end{bmatrix} = \begin{bmatrix} XE_c \\ I_n \\ K_c \end{bmatrix}, \quad Y_c = \mathcal{U}_c^{-*} \begin{bmatrix} I_n \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} I_n \\ -E_c^* X \\ 0 \end{bmatrix} \tag{13}$$

satisfy

$$\mathcal{E}_c U_c = Y_c E_c, \quad \mathcal{A}_c U_c = Y_c (A_c + B_c K_c),$$

i.e., U_c *is a basis matrix of a right reducing subspace of* $(\mathcal{E}_c, \mathcal{A}_c)$ *associated with* $(E_c, A_c + B_c K_c)$.

(vi) $\det R_c \neq 0$ *and the matrices* U_c, Y_c *defined in (13) satisfy*

$$U_c^* \mathcal{E}_c = (-E_c)^* Y_c^*, \quad U_c^* \mathcal{A}_c = (A_c + B_c K_c)^* Y_c^*,$$

i.e., U_c *is also a basis matrix of a left reducing subspace of* $(\mathcal{E}_c, \mathcal{A}_c)$ *associated with* $(-E_c^*, (A_c + B_c K_c)^*)$.

Moreover, if $X \in \mathbb{H}^{n,n}$ *solves the CARE (2), the matrix* K_c *in (iii) – (vi) is the same. It depends on* X *and has the expression*

$$K_c = -R_c^{-1}(B_c^* X E_c + N_c^*). \tag{14}$$

Proof. It is similar to the proof of Proposition 10. □

Notice that $\mathcal{E}_c = -\mathcal{E}_c^*$ and $\mathcal{A}_c = \mathcal{A}_c^*$. So the eigenvalues of $(\mathcal{E}_c, \mathcal{A}_c)$ appear in pairs $(\mu, -\bar{\mu})$, i.e., the spectrum has the *Hamiltonian structure*, e.g., [19, 26]. When the CARE (2) has an Hermitian solution, from (12) we have

$$\Lambda(\mathcal{E}_c, \mathcal{A}_c) = \Lambda(E_c, A_c + B_c K_c) \cup \Lambda(-E_c^*, (A_c + B_c K_c)^*) \cup \Lambda(0_p, I_p). \quad (15)$$

So the Hamiltonian structure is obvious. But in this case, if $\mu_0 \in \Lambda(\mathcal{E}_c, \mathcal{A}_c) \cap \mathbb{C}^0$, i.e., $\mu_0 = -\bar{\mu}_0$, μ_0 must be contained in both $\Lambda(E_c, A_c + B_c K_c)$ and $\Lambda(-E_c^*, (A_c + B_c K_c)^*)$ for any Hermitian solution X and its corresponding K_c. So the algebraic multiplicity of μ_0 (with respect to $(\mathcal{E}_c, \mathcal{A}_c)$) must be even. Moreover, for any Hermitian solution X and its corresponding K_c, from (15), $(E_c, A_c + B_c K_c)$ is regular if and only if $(\mathcal{E}_c, \mathcal{A}_c)$ is regular.

Sufficient conditions for the existence of Hermitian solutions of the AREs can be found in [19, 16, 8].

Finally, we review the Cayley transformation $\mathbf{c} : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$:

$$\mu = \mathbf{c}(\lambda) := (\lambda - 1)(\lambda + 1)^{-1}$$

with $\mathbf{c}(-1) = \infty$ and $\mathbf{c}(\infty) = 1$. The Cayley transformation is invertible and its inverse is

$$\lambda = \mathbf{c}^{-1}(\mu) = (1 + \mu)(1 - \mu)^{-1}.$$

The correspondence between λ and $\mu = \mathbf{c}(\lambda)$ is summarized in Table 1.

λ	$\lvert\lambda\rvert < 1$	$\lvert\lambda\rvert = 1$	$\lvert\lambda\rvert > 1$	1	0	-1	∞
μ	$\operatorname{Re}\mu < 0$	$\operatorname{Re}\mu = 0$	$\operatorname{Re}\mu > 0$	0	-1	∞	1

Table 1: Correspondence between λ and $\mu = \mathbf{c}(\lambda)$

The Cayley transformation can be generalized to the space $\mathbb{C}^{m,q} \times \mathbb{C}^{m,q}$. Let $(\mathcal{E}, \mathcal{A}) \in \mathbb{C}^{m,q} \times \mathbb{C}^{m,q}$. We can define

$$(\mathcal{F}, \mathcal{B}) = \mathbf{c}(\mathcal{E}, \mathcal{A}) =: (\mathcal{A} + \mathcal{E}, \mathcal{A} - \mathcal{E}) \in \mathbb{C}^{m,q} \times \mathbb{C}^{m,q}. \quad (16)$$

The eigenvalues of $(\mathcal{F}, \mathcal{B})$ and $(\mathcal{E}, \mathcal{A})$ are related by the scalar Cayley transformation, namely, $\lambda \in \Lambda(\mathcal{E}, \mathcal{A})$ if and only if $\mathbf{c}(\lambda) \in \Lambda(\mathcal{F}, \mathcal{B})$. Moreover, $\lambda, \mathbf{c}(\lambda)$ have the same Jordan structure.

3 Transformations between the AREs

We introduce the following transformation between the coefficient matrices of the DARE (1) and CARE (2). Given $(E_d, A_d, B_d, M_d, N_d, R_d)$, let $W \in \mathbb{C}^{n+p,n+p}$ be nonsingular such that $\begin{bmatrix} A_d + E_d & B_d \end{bmatrix} W^{-1} = \begin{bmatrix} H & 0 \end{bmatrix}$, where $H \in \mathbb{C}^{n,n}$. Then we define

$$\mathbf{f}_W : (E_d, A_d, B_d, M_d, N_d, R_d) \mapsto (E_c, A_c, B_c, M_c, N_c, R_c),$$

where $E_c, A_c, B_c, M_c, N_c, R_c$ satisfy

$$\begin{bmatrix} E_c & 0 \\ A_c & B_c \end{bmatrix} = \mathcal{X} \begin{bmatrix} E_d & 0 \\ A_d & B_d \end{bmatrix} W^{-1} = \frac{\sqrt{2}}{2} \begin{bmatrix} A_d + E_d & B_d \\ A_d - E_d & B_d \end{bmatrix} W^{-1}, \quad (17)$$

$$\begin{bmatrix} M_c & N_c \\ N_c^* & R_c \end{bmatrix} = W^{-*} \begin{bmatrix} M_d & N_d \\ N_d^* & R_d \end{bmatrix} W^{-1} \quad (18)$$

with

$$\mathcal{X} = \frac{\sqrt{2}}{2} \begin{bmatrix} I_n & I_n \\ -I_n & I_n \end{bmatrix}. \quad (19)$$

Note that (17) can be considered as an LU or LQ factorization of $\mathcal{X} \left[\begin{smallmatrix} E_d & 0 \\ A_d & B_d \end{smallmatrix}\right]$ ([10]). So W depends on (E_d, A_d, B_d), but it always exists and can be chosen unitary. Note also that $\mathbf{f}_W$ depends on W. But once W has been chosen, the transformation $\mathbf{f}_W$ is uniquely determined.

Similarly, given a tuple $(E_c, A_c, B_c, M_c, N_c, R_c)$, for any nonsingular matrix $\tilde{W} \in \mathbb{C}^{n+p,n+p}$ satisfying $\begin{bmatrix} E_c + A_c & -B_c \end{bmatrix} \tilde{W} = \begin{bmatrix} \tilde{H} & 0 \end{bmatrix}$ with $\tilde{H} \in \mathbb{C}^{n,n}$, we can define

$$\tilde{\mathbf{f}}_{\tilde{W}} : (E_c, A_c, B_c, M_c, N_c, R_c) \mapsto (E_d, A_d, B_d, M_d, N_d, R_d),$$

where $E_d, A_d, B_d, M_d, N_d, R_d$ satisfy

$$\begin{bmatrix} E_d & 0 \\ A_d & B_d \end{bmatrix} = \mathcal{X}^* \begin{bmatrix} E_c & 0 \\ A_c & B_c \end{bmatrix} \tilde{W} = \frac{\sqrt{2}}{2} \begin{bmatrix} E_c - A_c & -B_c \\ E_c + A_c & B_c \end{bmatrix} \tilde{W}, \quad (20)$$

$$\begin{bmatrix} M_d & N_d \\ N_d^* & R_d \end{bmatrix} = \tilde{W}^* \begin{bmatrix} M_c & N_c \\ N_c^* & R_c \end{bmatrix} \tilde{W}. \quad (21)$$

For a specific pair of tuples satisfying (17) and (18) with a fixed W, we have

$$\begin{bmatrix} E_d & 0 \\ A_d & B_d \end{bmatrix} = \mathcal{X}^* \begin{bmatrix} E_c & 0 \\ A_c & B_c \end{bmatrix} W = \frac{\sqrt{2}}{2} \begin{bmatrix} E_c - A_c & -B_c \\ E_c + A_c & B_c \end{bmatrix} W,$$

$$\begin{bmatrix} M_d & N_d \\ N_d^* & R_d \end{bmatrix} = W^* \begin{bmatrix} M_c & N_c \\ N_c^* & R_c \end{bmatrix} W.$$

So $(E_d, A_d, B_d, M_d, N_d, R_d)$ can be recovered from $(E_c, A_c, B_c, M_c, N_c, R_c)$ by the transformation $\tilde{\mathbf{f}}_{\tilde{W}}$ with $\tilde{W} = W$. In this case, $\tilde{\mathbf{f}}_W$ behaves as an inverse operation of $\mathbf{f}_W$. For this reason, from now on we will abuse the notations by replacing $\tilde{W}$ with W in (20) and (21) and $\tilde{\mathbf{f}}_{\tilde{W}}$ with $\mathbf{f}_W^{-1}$, the "inverse" of $\mathbf{f}_W$. This should not cause any confusion, since in the following we will consider either the transformations $\mathbf{f}_W$ and $\tilde{\mathbf{f}}_{\tilde{W}}$ alone or a specific pair of tuples $(E_d, A_d, B_d, M_d, N_d, R_d)$ and $(E_c, A_c, B_c, M_c, N_c, R_c)$ related by (17) and (18) with a fixed W.

From (18) and (21), it is easily seen that the Hermitian matrices $\begin{bmatrix} M_d & N_d \\ N_d^* & R_d \end{bmatrix}$ and $\begin{bmatrix} M_c & N_c \\ N_c^* & R_c \end{bmatrix}$ have the same inertia indices under the transformation $\mathbf{f}_W$. The matrix triplets (E_d, A_d, B_d) and (E_c, A_c, B_c) have the following relations.

Theorem 12 *Suppose that the matrix triplets (E_d, A_d, B_d) and (E_c, A_c, B_c) satisfy (17) or (20).*

(a) *(E_d, A_d, B_d) is controllable at $\lambda \in \mathbb{C}$ ($\lambda \neq -1$) if and only if (E_c, A_c, B_c) is controllable at $\mu = \mathbf{c}(\lambda) \in \mathbb{C}$ ($\mu \neq 1$).*

(b) *(E_d, A_d, B_d) is controllable at -1 if and only if $\det E_c \neq 0$.*

(c) *$\det E_d \neq 0$ if and only if (E_c, A_c, B_c) is controllable at 1.*

(d) *$\det E_d \neq 0$ and (E_d, A_d, B_d) is D-stabilizable if and only if $\det E_c \neq 0$ and (E_c, A_c, B_c) is C-stabilizable.*

(e) *$\det E_d \neq 0$ and (E_d, A_d, B_d) is controllable if and only if $\det E_c \neq 0$ and (E_c, A_c, B_c) is controllable.*

(f) *(E_d, A_d, B_d) is regularizable if and only if (E_c, A_c, B_c) is regularizable.*

Proof. Pre-multiplying $[-\lambda I_n, I_n]$ to (20), simple calculations yield

$$[A_d - \lambda E_d, B_d] = \frac{\sqrt{2}}{2}[(\lambda+1)A_c - (\lambda-1)E_c, (\lambda+1)B_c]W. \tag{22}$$

Similarly, pre-multiplying $[-\mu I_n, I_n]$ to (17) we have

$$[A_c - \mu E_c, B_c] = \frac{\sqrt{2}}{2}[(1-\mu)A_d - (1+\mu)E_d, (1-\mu)B_d]W^{-1}. \tag{23}$$

(a) For any $\lambda \in \mathbb{C}$ such that $\lambda \neq -1$, we have $\lambda + 1 \neq 0$. Then (22) can be written as

$$[A_d - \lambda E_d, B_d] = \frac{\sqrt{2}}{2}(\lambda + 1)[A_c - \mu E_c, B_c]W$$

where $\mu = \mathbf{c}(\lambda) = (\lambda - 1)(\lambda + 1)^{-1}$. Clearly, $\mu \neq 1, \infty$ and

$$\operatorname{rank}[A_d - \lambda E_d, B_d] = n \Longleftrightarrow \operatorname{rank}[A_c - \mu E_c, B_c] = n.$$

(b) When $\lambda = -1$, (22) becomes

$$[A_d + E_d, B_d] = \sqrt{2}[E_c, 0]W.$$

So

$$\operatorname{rank}[A_d + E_d, B_d] = n \Longleftrightarrow \operatorname{rank} E_c = n.$$

(c) It can be obtained by using (23).

(d) From (c), $\det E_d \neq 0$ is equivalent to (E_c, A_c, B_c) being controllable at 1. From (b), (E_d, A_d, B_d) being controllable at -1 is equivalent to $\det E_c \neq 0$. From (a) and the relation between λ and $\mu = \mathbf{c}(\lambda)$ shown in Table 1, (E_d, A_d, B_d) being controllable at any $\lambda \in \mathbb{O}^0 \cup \mathbb{O}^+$ with $\lambda \neq -1$ is equivalent to (E_c, A_c, B_c) being controllable at $\mu = \mathbf{c}(\lambda) \in \mathbb{C}^0 \cup \mathbb{C}^+$ with $\mu \neq 1$. The result follows from all these equivalences.

(e) analogous to **(d)**.

(f) If (E_d, A_d, B_d) is regularizable, there exists $\lambda \in \mathbb{C}$ such that $\operatorname{rank}[A_d - \lambda E_d,\ B_d] = n$. If $\lambda \neq -1$, by (a), $\operatorname{rank}[A_c - \mathbf{c}(\lambda) E_c,\ B_c] = n$. So (E_c, A_c, B_c) is regularizable. If $\lambda = -1$, by (b), $\det E_c \neq 0$. Then $\operatorname{rank}[A_c - \mu E_c,\ B_c] = n$ for some $\mu \notin \Lambda(E_c, A_c)$. Again, (E_c, A_c, B_c) is regularizable. Similarly, one can show that when (E_c, A_c, B_c) is regularizable, (E_d, A_d, B_d) is also regularizable. □

Unfortunately, when (E_d, A_d, B_d) is controllable at infinity, the corresponding triplet (E_c, A_c, B_c) is not necessarily controllable at $1 = \mathbf{c}(\infty)$. From Theorem 12 (c), it is true only when $\det E_d \neq 0$. Similarly, only when $\det E_c \neq 0$, (E_c, A_c, B_c) being controllable at infinity implies (E_d, A_d, B_c) being controllable at $-1 = \mathbf{c}^{-1}(\infty)$.

Since the matrix pair $(\mathcal{E}_d, \mathcal{A}_d)$ in (5) is uniquely determined by the matrix tuple $(E_d, A_d, B_d, M_d, N_d, R_d)$ and $(\mathcal{E}_c, \mathcal{A}_c)$ is uniquely determined by the matrix tuple $(E_c, A_c, B_c, M_c, N_c, R_c)$, the transformation $\mathbf{f}_W$ and its inverse can be considered the transformations between the matrix pairs $(\mathcal{E}_d, \mathcal{A}_d)$ and $(\mathcal{E}_c, \mathcal{A}_c)$:

$$(\mathcal{E}_c, \mathcal{A}_c) = \mathbf{f}_W(\mathcal{E}_d, \mathcal{A}_d), \qquad (\mathcal{E}_d, \mathcal{A}_d) = \mathbf{f}_W^{-1}(\mathcal{E}_c, \mathcal{A}_c),$$

with the blocks determined by the formulas (17) – (18) and (20) – (21), respectively. Since $R_d + B_d^* X B_d$ in (1) and R_c in (2) are required to be invertible, not every tuple $(E_d, A_d, B_d, M_d, N_d, R_d)$ corresponds to a DARE. Similarly, not every $(E_c, A_c, B_c, M_c, N_c, R_c)$ corresponds to a CARE. However, if we consider the inverses as formal symbols, we may also consider $\mathbf{f}_W$ and $\mathbf{f}_W^{-1}$ as transformations defined on the ARE operators:

$$\mathbf{R}_c(X) = \mathbf{f}_W(\mathbf{R}_d(X)), \qquad \mathbf{R}_d(X) = \mathbf{f}_W^{-1}(\mathbf{R}_c(X)).$$

In [26], a transformation $\mathbf{t}$ was introduced between the matrix pairs of the forms

$$\begin{aligned} (\mathcal{F}_d, \mathcal{G}_d) &= \left(\begin{bmatrix} 0 & F_d \\ -G_d^* & 0 \end{bmatrix}, \begin{bmatrix} 0 & G_d \\ -F_d^* & D_d \end{bmatrix} \right), \\ (\mathcal{F}_c, \mathcal{G}_c) &= \left(\begin{bmatrix} 0 & F_c \\ -F_c^* & 0 \end{bmatrix}, \begin{bmatrix} 0 & G_c \\ G_c^* & D_c \end{bmatrix} \right). \end{aligned}$$

The transformation $\mathbf{t}$ is defined by

$$(\mathcal{F}_c, \mathcal{G}_c) = \mathbf{t}(\mathcal{F}_d, \mathcal{G}_d), \qquad F_c = G_d + F_d, \quad G_c = G_d - F_d, \quad D_c = D_d,$$

and its inverse is

$$(\mathcal{F}_d, \mathcal{G}_d) = \mathbf{t}^{-1}(\mathcal{F}_c, \mathcal{G}_c), \qquad F_d = \frac{1}{2}(F_c - G_c), \quad G_d = \frac{1}{2}(F_c + G_c), \quad D_d = D_c.$$

The transformation $\mathbf{t}$ can be considered as an intermediate transformation of $\mathbf{f}_W$. In fact, the pair $(\mathcal{E}_d, \mathcal{A}_d)$ has the form of $(\mathcal{F}_d, \mathcal{G}_d)$ with

$$F_d = \begin{bmatrix} E_d & 0 \end{bmatrix}, \quad G_d = \begin{bmatrix} A_d & B_d \end{bmatrix}, \quad D_d = \begin{bmatrix} M_d & N_d \\ N_d^* & R_d \end{bmatrix},$$

and the pair $(\mathcal{E}_c, \mathcal{A}_c)$ has the form of $(\mathcal{F}_c, \mathcal{G}_c)$ with

$$F_c = \begin{bmatrix} E_c & 0 \end{bmatrix}, \quad G_c = \begin{bmatrix} A_c & B_c \end{bmatrix}, \quad D_c = \begin{bmatrix} M_c & N_c \\ N_c^* & R_c \end{bmatrix}.$$

Applying $\mathbf{t}$ and $\mathbf{t}^{-1}$ to $(\mathcal{E}_d, \mathcal{A}_d)$ and $(\mathcal{E}_c, \mathcal{A}_c)$, respectively, we have

$$(\tilde{\mathcal{E}}_c, \tilde{\mathcal{A}}_c) = \mathbf{t}(\mathcal{E}_d, \mathcal{A}_d) = \left(\begin{bmatrix} 0 & A_d + E_d & B_d \\ -(A_d + E_d)^* & 0 & 0 \\ -B_d^* & 0 & 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 0 & A_d - E_d & B_d \\ (A_d - E_d)^* & M_d & N_d \\ B_d^* & N_d^* & R_d \end{bmatrix} \right), \tag{24}$$
$$(\tilde{\mathcal{E}}_d, \tilde{\mathcal{A}}_d) = \mathbf{t}^{-1}(\mathcal{E}_c \, \mathcal{A}_c) = \left(\begin{bmatrix} 0 & \frac{1}{2}(E_c - A_c) & -\frac{1}{2}B_c \\ -\frac{1}{2}(E_c + A_c)^* & 0 & 0 \\ -\frac{1}{2}B_c^* & 0 & 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 0 & \frac{1}{2}(E_c + A_c) & \frac{1}{2}B_c \\ -\frac{1}{2}(E_c - A_c)^* & M_c & N_c \\ \frac{1}{2}B_c^* & N_c^* & R_c \end{bmatrix} \right). \tag{25}$$

Let $\mathcal{W} = \operatorname{diag}(\sqrt{2}I_n, W)$, where W satisfies (17). Then for $(\mathcal{E}_c, \mathcal{A}_c) = \mathbf{f}_W(\mathcal{E}_d, \mathcal{A}_d)$, we have

$$(\mathcal{E}_c, \mathcal{A}_c) = (\mathcal{W}^{-*}\tilde{\mathcal{E}}_c\mathcal{W}^{-1},\, \mathcal{W}^{-*}\tilde{\mathcal{A}}_c\mathcal{W}^{-1}), \tag{26}$$

i.e., $(\mathcal{E}_c, \mathcal{A}_c)$ is equivalent to $(\tilde{\mathcal{E}}_c, \tilde{\mathcal{A}}_c)$. Similarly, for $(\mathcal{E}_d, \mathcal{A}_d) = \mathbf{f}_W^{-1}(\mathcal{E}_c, \mathcal{A}_c)$, we have

$$(\mathcal{E}_d, \mathcal{A}_d) = (\mathcal{W}^*\tilde{\mathcal{E}}_d\mathcal{W},\, \mathcal{W}^*\tilde{\mathcal{A}}_d\mathcal{W}),$$

i.e., $(\mathcal{E}_d, \mathcal{A}_d)$ is equivalent to $(\tilde{\mathcal{E}}_d, \tilde{\mathcal{A}}_d)$.

Below, we will also give relations between the ARE solutions and the reducing subspaces of $(\tilde{\mathcal{E}}_d, \tilde{\mathcal{A}}_d)$ and $(\tilde{\mathcal{E}}_c, \tilde{\mathcal{A}}_c)$. Since $\mathbf{t}$ is simpler than $\mathbf{f}_W$, numerically, one may use the transformation $\mathbf{t}$ instead of $\mathbf{f}_W$.

In the following two sections we will study the relation between the AREs under the transformation $\mathbf{f}_W$.

4 Transforming a DARE to a CARE

In this section we assume that the CARE (2) and $(\mathcal{E}_c, \mathcal{A}_c)$ are transformed from the DARE (1) and $(\mathcal{E}_d, \mathcal{A}_d)$, respectively, by the transformation $\mathbf{f}_W$ defined by (17) and (18).

The following theorem gives the existence condition for the CARE and the relation between the Hermitian solutions of both AREs.

Theorem 13 *Consider the DARE (1) and the CARE (2), where* $\mathbf{R}_c(X) = \mathbf{f}_W(\mathbf{R}_d(X))$.

(a) *If* $\det R_c \neq 0$, *then every* $X \in \mathbb{H}^{n,n}$ *that solves the DARE (1) also solves the CARE (2).*

(b) *If* $\det R_c = 0$, *then* $\mathbf{R}_c(X)$ *is not defined.*

Proof. For the matrix $\mathcal{X}$ defined in (19) we have

$$\begin{bmatrix} -X & 0 \\ 0 & X \end{bmatrix} = \mathcal{X}^* \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \mathcal{X}.$$

Then

$$\begin{aligned} \mathbf{D}_d(X) &= \begin{bmatrix} M_d & N_d \\ N_d^* & R_d \end{bmatrix} + \begin{bmatrix} E_d & 0 \\ A_d & B_d \end{bmatrix}^* \begin{bmatrix} -X & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} E_d & 0 \\ A_d & B_d \end{bmatrix} \\ &= \begin{bmatrix} M_d & N_d \\ N_d^* & R_d \end{bmatrix} + \begin{bmatrix} E_d & 0 \\ A_d & B_d \end{bmatrix}^* \mathcal{X}^* \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \mathcal{X} \begin{bmatrix} E_d & 0 \\ A_d & B_d \end{bmatrix} \\ &= W^* \left(\begin{bmatrix} M_c & N_c \\ N_c^* & R_c \end{bmatrix} + \begin{bmatrix} E_c & 0 \\ A_c & B_c \end{bmatrix}^* \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \begin{bmatrix} E_c & 0 \\ A_c & B_c \end{bmatrix} \right) W \\ &= W^* \mathbf{D}_c(X) W. \end{aligned}$$

(a) Suppose $X \in \mathbb{H}^{n,n}$ solves the DARE (1). By Proposition 10 (iii),

$$\operatorname{rank} \mathbf{D}_c(X) = \operatorname{rank} \mathbf{D}_d(X) = p, \tag{27}$$

and for

$$S = W \begin{bmatrix} I_n \\ K_d \end{bmatrix} =: \begin{matrix} & n \\ n & \\ p & \end{matrix}\!\!\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}, \tag{28}$$

where K_d is of the form (10), we have

$$\mathbf{D}_c(X) S = W^{-*} \mathbf{D}_d(X) \begin{bmatrix} I_n \\ K_d \end{bmatrix} = 0. \tag{29}$$

If $\det S_1 \neq 0$, then

$$\mathbf{D}_c(X) \begin{bmatrix} I_n \\ S_2 S_1^{-1} \end{bmatrix} = 0.$$

Then by Proposition 11 (iii), X solves the CARE (2). Due to (27), (29), and Lemma 9, $\det S_1 \neq 0$ if and only if $\det R_c \neq 0$.

(b) It is obvious. □

When $\det R_c \neq 0$, the relation between a DARE solution X and a reducing subspaces of $(\mathcal{E}_c, \mathcal{A}_c)$ is described by the following theorem.

Theorem 14 *Consider the DARE (1) and the CARE (2), where* $\mathbf{R}_c(X) = \mathbf{f}_W(\mathbf{R}_d(X))$. *Let* $(\mathcal{E}_c, \mathcal{A}_c)$ *be of the form (6). Suppose* $X \in \mathbb{H}^{n,n}$ *is a solution of the DARE (1) and the corresponding* K_d *is of the form (10), and suppose* $\det R_c \neq 0$. *Then we have the following results.*

(i) *The matrices*

$$U_c = \begin{bmatrix} XE_c \\ I_n \\ K_c \end{bmatrix}, \quad Y_c = \begin{bmatrix} I_n \\ -E_c^* X \\ 0 \end{bmatrix}$$

satisfy

$$\mathcal{E}_c U_c = Y_c E_c, \quad \mathcal{A}_c U_c = Y_c(A_c + B_c K_c),$$

as well as

$$U_c^* \mathcal{E}_c = -E_c^* Y_c^*, \quad U_c^* \mathcal{A}_c = (A_c + B_c K_c)^* Y_c^*,$$

where K_c *is of the form (14).*

(ii) *The relations between* K_d *and* K_c, $(E_d, A_d + B_d K_d)$ *and* $(E_c, A_c + B_c K_c)$ *are given, respectively, by*

$$\begin{bmatrix} I_n \\ K_c \end{bmatrix} = S S_1^{-1} = W \begin{bmatrix} I_n \\ K_d \end{bmatrix} S_1^{-1}, \tag{30}$$

and

$$(E_c, A_c + B_c K_c) = \mathbf{c}\left(\frac{\sqrt{2}}{2} E_d S_1^{-1}, \frac{\sqrt{2}}{2}(A_d + B_d K_d) S_1^{-1} \right), \tag{31}$$

where S, S_1 *are defined in (28), and* $\mathbf{c}$ *is the Cayley transformation (16).*

Moreover, if X *is a stabilizing (resp. semi-stabilizing) solution of the DARE (1), i.e.,* $(E_d, A_d + B_d K_d)$ *is D-stable (resp. D-semi-stable), then* X *is also a stabilizing (resp. semi-stabilizing) solution of the CARE (2), i.e.,* $(E_c, A_c + B_c K_c)$ *is C-stable (resp. C-semi-stable).*

Proof. Part (i) follows simply from Theorem 13 and Proposition 11 (v). For part (ii), since $\det S_1 \neq 0$, (30) follows from (29). By (17) and (28),

$$\begin{aligned}
&\begin{bmatrix} E_c \\ A_c + B_c K_c \end{bmatrix} = \begin{bmatrix} E_c & 0 \\ A_c & B_c \end{bmatrix} \begin{bmatrix} I_n \\ K_c \end{bmatrix} \\
&= \mathcal{X} \begin{bmatrix} E_d & 0 \\ A_d & B_d \end{bmatrix} W^{-1} W \begin{bmatrix} I_n \\ K_d \end{bmatrix} S_1^{-1} = \frac{\sqrt{2}}{2} \begin{bmatrix} A_d + E_d & B_d \\ A_d - E_d & B_d \end{bmatrix} \begin{bmatrix} I_n \\ K_d \end{bmatrix} S_1^{-1} \\
&= \frac{\sqrt{2}}{2} \begin{bmatrix} A_d + B_d K_d + E_d \\ A_d + B_d K_d - E_d \end{bmatrix} S_1^{-1}.
\end{aligned}$$

So we have (31).

The last statement follows from (31) and the property of the Cayley transformation $\mathbf{c}$. □

The following example shows that $\det R_c = 0$ may occur although the DARE has Hermitian solutions.

Example 1 Consider the DARE

$$X - (-1)X(-1) - (X + a)(b + X)^{-1}(X + a) = 0,$$

where $a, b \in \mathbb{R}$ and $a \neq b$. The coefficient matrices are $A_d = B_d = 1$, $E_d = -1$, $M_d = 0$, $N_d = a$, $R_d = b$. The DARE has a unique solution $X = -a$. It is easily verified that any nonsingular matrix W satisfying

$$\begin{bmatrix} E_c & 0 \\ A_c & B_c \end{bmatrix} = \mathcal{X} \begin{bmatrix} E_d & 0 \\ A_d & B_d \end{bmatrix} W^{-1} = \frac{\sqrt{2}}{2} \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} W^{-1}$$

has the form $W = \begin{bmatrix} 0 & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$, where $w_{12}w_{21} \neq 0$. By (18), it is easily verified that $R_c = 0$ for all W of the above form.

In the following we give some sufficient conditions for $\det R_c \neq 0$.

Theorem 15 *Consider the DARE (1) and the CARE (2), where* $\mathbf{R}_c(X) = \mathbf{f}_W(\mathbf{R}_d(X))$. *Suppose that the DARE has (at least) one Hermitian solution. Let* $(\mathcal{E}_d, \mathcal{A}_d)$ *be of the form (5).*

(a) *If* $\det(\mathcal{E}_d + \mathcal{A}_d) \neq 0$, *then* $\det R_c \neq 0$ *for any* W *satisfying (17).*

(b) *If* $\det(\mathcal{E}_d + \mathcal{A}_d) = 0$ *and* (E_d, A_d, B_d) *is controllable at* -1, *then* $\det R_c = 0$ *for all* W *satisfying (17).*

Proof. From the proof of Theorem 13, $\det R_c \neq 0$ if and only if $\det S_1 \neq 0$. So we only need to consider $\det S_1$.

Partition

$$W = \begin{matrix} & n & p \\ n & W_{11} & W_{12} \\ p & W_{21} & W_{22} \end{matrix}, \quad W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}.$$

Let $X \in \mathbb{H}^{n,n}$ be a solution to the DARE and K_d be of the form (10). Then

$$S_1 = W_{11} + W_{12} K_d.$$

From (17),

$$A_d + E_d = \sqrt{2} E_c W_{11}, \quad B_d = \sqrt{2} E_c W_{12}.$$

Hence

$$E_d + A_d + B_d K_d = \sqrt{2} E_c (W_{11} + W_{12} K_d) = \sqrt{2} E_c S_1. \tag{32}$$

(a) When $\det(\mathcal{E}_d + \mathcal{A}_d) \neq 0$, i.e., $-1 \notin \Lambda(\mathcal{E}_d, \mathcal{A}_d)$, by (11), $\det(E_d + A_d + B_d K_d) \neq 0$. Then S_1, as well as E_c, is nonsingular.

(b) Since (E_d, A_d, B_d) is controllable at -1, by Theorem 12 (b), $\det E_c \neq 0$. On the other hand $\det(\mathcal{E}_d + \mathcal{A}_d) = 0$ implies $\det(E_d + A_d + B_d K_d) = 0$. From (32), we have $\det S_1 = 0$. □

The following theorem gives an equivalent condition for $\det(\mathcal{E}_d + \mathcal{A}_d) \neq 0$.

Theorem 16 *Consider the pair $(\mathcal{E}_d, \mathcal{A}_d)$ of the form (5). Let L_d be a basis matrix of* $\text{null}[A_d + E_d,\ B_d]$*. Then* $\det(\mathcal{E}_d + \mathcal{A}_d) \neq 0$ *if and only if* (E_d, A_d, B_d) *is controllable at -1 and $L_d^* \begin{bmatrix} M_d & N_d \\ N_d^* & R_d \end{bmatrix} L_d$ is nonsingular.*

Proof. See [26]. □

Remark 1 The condition $\det(\mathcal{E}_d + \mathcal{A}_d) \neq 0$ implies that $(\mathcal{E}_d, \mathcal{A}_d)$ is regular and $-1 \notin \Lambda(\mathcal{E}_d, \mathcal{A}_d)$. By (11) this also implies that $(E_d, A_d + B_d K_d)$ is regular and $-1 \notin \Lambda(E_d, A_d + B_d K_d)$ for any Hermitian solution X. Also, Theorem 16 shows that $\det(\mathcal{E}_d + \mathcal{A}_d) \neq 0$ implies (E_d, A_d, B_d) is controllable at -1.

The condition $\det(\mathcal{E}_d + \mathcal{A}_d) = 0$ implies $-1 \in \Lambda(\mathcal{E}_d, \mathcal{A}_d)$ and/or $(\mathcal{E}_d, \mathcal{A}_d)$ is singular. Theorem 16 shows that this happens if $L_d^* \begin{bmatrix} M_d & N_d \\ N_d^* & R_d \end{bmatrix} L_d$ is singular and/or (E_d, A_d, B_d) is not controllable at -1.

The situation is complicated when $\det(\mathcal{E}_d + \mathcal{A}_d) = 0$ and (E_d, A_d, B_d) is not controllable at -1. In this case we turn to consider a DARE that is reduced from (1) with the decomposition (4). When (E_d, A_d, B_d) is not controllable at -1, by Proposition 8 (ii) (with $\Omega = \{-1\}$), there are unitary matrices P_d, Q_d such that

$$E_d = P_d \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix} Q_d,\ A_d = P_d \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} Q_d,\ B_d = P_d \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \quad (33)$$

with (E_{11}, A_{11}, B_1) controllable at -1 and $\Lambda(E_{22}, A_{22}) \subseteq \{-1\}$.

Partition

$$\begin{aligned} Q_d M_d Q_d^* &= \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{bmatrix}, \qquad Q_d N_d = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}, \\ P_d^* X P_d &= \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}, \qquad K_d Q_d^* = [K_1,\ K_2] \end{aligned} \quad (34)$$

conformably, where $X \in \mathbb{H}^{n,n}$ and K_d is of the form (10). It is easily verified that if X solves the DARE (1), then X_{11} solves the reduced DARE associated with the coefficient matrices $(E_{11}, A_{11}, B_1, M_{11}, N_1, R_d)$, and K_1 defined in (34) has the form

$$K_1 = -(R_d + B_1^* X_{11} B_1)^{-1} (B_1^* X_{11} A_{11} + N_1^*).$$

The reduced DARE has the associated matrix pair

$$(\hat{\mathcal{E}}_d, \hat{\mathcal{A}}_d) = \left(\begin{bmatrix} 0 & E_{11} & 0 \\ -A_{11}^* & 0 & 0 \\ -B_1^* & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{11} & B_1 \\ -E_{11}^* & M_{11} & N_1 \\ 0 & N_1^* & R_d \end{bmatrix} \right).$$

Let $T = [T_{ij}]_{2\times 2}$ be nonsingular and satisfy

$$\frac{\sqrt{2}}{2} \begin{bmatrix} A_{11} + E_{11} & B_1 \\ A_{11} - E_{11} & B_1 \end{bmatrix} T^{-1} = \begin{bmatrix} \hat{E}_{11} & 0 \\ \hat{A}_{11} & \hat{B}_1 \end{bmatrix}. \quad (35)$$

Define

$$W_d = \left[\begin{array}{cc|c} T_{11} & 0 & T_{12} \\ 0 & I & 0 \\ \hline T_{21} & 0 & T_{22} \end{array}\right] \begin{bmatrix} Q_d & 0 \\ 0 & I_p \end{bmatrix} =: \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}. \quad (36)$$

Then W_d is nonsingular and satisfies (17). So W_d defines a transformation $\mathbf{f}_{W_d}$.

Theorem 17 *Consider the DARE (1). Suppose that* (A_d, B_d, E_d) *is not controllable at* -1 *and has the condensed form (33), and the DARE has an Hermitian solution. Consider the CARE*

$$\mathbf{R}_c(X) = \mathbf{f}_{W_d}(\mathbf{R}_d(X)) = 0,$$

where W_d *is defined in (36). If* $\det(\hat{\mathcal{E}}_d + \hat{\mathcal{A}}_d) \neq 0$*, then the matrix* R_c *defined in (18) with* $W = W_d$ *is nonsingular.*

Proof. Once again we only need to prove $\det S_1 \neq 0$, where S_1 is the top block of S defined in (28) with $W = W_d$.

By using the block forms of W_d and K_d, we have

$$S_1 = W_{11} + W_{12}K_d = \begin{bmatrix} T_{11} + T_{12}K_1 & T_{12}K_2 \\ 0 & I \end{bmatrix} Q_d.$$

So $\det S_1 \neq 0$ when $\det(T_{11} + T_{12}K_1) \neq 0$. The latter can be obtained by applying Theorem 15 (a) to the reduced DARE with the transformation $\mathbf{f}_T$, where T is defined in (35). □

In Theorem 17, the non-singularity of R_c depends on the choice of W.

When (E_d, A_d, B_d) is not controllable at -1 and $\det(\hat{\mathcal{E}}_d + \hat{\mathcal{A}}_d) = 0$, R_c may or may not be nonsingular.

Example 2 Consider the DARE with coefficient matrices

$$A_d = I_2,\ E_d = -I_2,\ B_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ M_d = \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix},\ N_d = \begin{bmatrix} a \\ e \end{bmatrix},\ R_d = b,$$

where $a, b, c, e \in \mathbb{R}$ and $a \neq b$, $c(b-a) \geq 0$. The general form of Hermitian solutions to the DARE is

$$X = \begin{bmatrix} -a & e^{i\alpha}\sqrt{c(b-a)} - e \\ e^{-i\alpha}\sqrt{c(b-a)} - e & x_{22} \end{bmatrix},$$

where $\alpha, x_{22} \in \mathbb{R}$.

The matrix triplet (E_d, A_d, B_d) is not controllable at -1 and it is already in the condensed form (33) with $A_{11} = -E_{11} = B_1 = 1$, and $M_{11} = 0$, $N_1 = a$, $R_c = b$. It is easily verified $\det(\hat{\mathcal{E}}_d + \hat{\mathcal{A}}_d) = 0$. On the other hand, any matrix W^{-1} satisfying (17) has the form

$$W^{-1} = \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & 0 \end{bmatrix}.$$

So we have

$$R_c = \begin{bmatrix} w_{13} \\ w_{23} \\ 0 \end{bmatrix}^* \left[\begin{array}{cc|c} 0 & 0 & a \\ 0 & c & e \\ \hline a & e & b \end{array}\right] \begin{bmatrix} w_{13} \\ w_{23} \\ 0 \end{bmatrix} = c|w_{23}|^2.$$

When $c \neq 0$, for those W^{-1} with $w_{23} \neq 0$ (which clearly exist), we have $R_c \neq 0$. When $c = 0$, for all W^{-1} we have $R_c = 0$.

Although $\mathbf{f}_W$ may fail to transform a DARE to a CARE, the Hermitian solutions of a DARE can always be related to a reducing subspace of the matrix pair $(\tilde{\mathcal{E}}_c, \tilde{\mathcal{A}}_c) = \mathbf{t}(\mathcal{E}_d, \mathcal{A}_d)$ defined in (24).

Theorem 18 *Consider the DARE (1). Let* $(\tilde{\mathcal{E}}_c, \tilde{\mathcal{A}}_c) = \mathbf{t}(\mathcal{E}_d, \mathcal{A}_d)$*, which is defined in (24). Then* $X \in \mathbb{H}^{n,n}$ *solves the DARE if and only if* $\det(R_d + B_d^* X B_d) \neq 0$ *and the matrices*

$$\tilde{U}_c = \begin{bmatrix} X(E_d + A_d + B_d K_d) \\ 2I_n \\ 2K_d \end{bmatrix}, \quad \tilde{Y}_c = \begin{bmatrix} 2I_n \\ -(A_d + E_d)^* X \\ -B_d^* X \end{bmatrix}$$

satisfy

$$\tilde{\mathcal{E}}_c \tilde{U}_c = \tilde{Y}_c (A_d + B_d K_d + E_d), \quad \tilde{\mathcal{A}}_c \tilde{U}_c = \tilde{Y}_c (A_d + B_d K_d - E_d), \tag{37}$$

as well as

$$\tilde{U}_c^* \tilde{\mathcal{E}}_c = -(A_d + B_d K_d + E_d)^* \tilde{Y}_c^*, \quad \tilde{U}_c^* \tilde{\mathcal{A}}_c = (A_d + B_d K_d - E_d)^* \tilde{Y}_c^*,$$

where K_d *is of the form (10).*

Proof. The relations in (37) can be verified directly. The last two relations follow from the first two by taking conjugate transpose. □

This result may be useful for numerically solving the DARE.

The result also has some interesting properties. The first property is that for U_d, V_d in (8) and Y_d, Z_d in (9), we have

$$\tilde{U}_c = U_d + V_d, \qquad \tilde{Y}_c = Y_d + Z_d.$$

The second property is that the sub-pairs associated with $\tilde{U}_c$ and U_d are related by the Cayley transformation

$$(A_d + B_d K_d + E_d,\ A_d + B_d K_d - E_d) = \mathbf{c}(E_d,\ A_d + B_d K_d).$$

The third property is that Theorem 18 actually is a generalization of Theorem 14. In fact, by (26) and (37) we have

$$\mathcal{E}_c(\mathcal{W}\tilde{U}_c) = (\mathcal{W}^{-*}\tilde{Y}_c)(A_d+B_dK_d+E_d), \quad \mathcal{A}_c(\mathcal{W}\tilde{U}_c) = (\mathcal{W}^{-*}\tilde{Y}_c)(A_d+B_dK_d-E_d),$$

where $\mathcal{W} = \mathrm{diag}(\sqrt{2}I, W)$. By (28) and (17),

$$\mathcal{W}\tilde{U}_c = \begin{bmatrix} \frac{\sqrt{2}}{2}X(E_d + A_d + B_dK_d) \\ S_1 \\ S_2 \end{bmatrix}, \qquad \mathcal{W}^{-*}\tilde{Y}_c = \frac{\sqrt{2}}{2}\begin{bmatrix} I_n \\ -E_cX \\ 0 \end{bmatrix}.$$

When $\det S_1 \neq 0$, we have Theorem 14 again.

Finally, when $(\mathcal{E}_d, \mathcal{A}_d)$ is a real pair, if the matrix W is chosen to be real, the pair $(\mathcal{E}_c, \mathcal{A}_c) = \mathbf{f}_W(\mathcal{E}_d, \mathcal{A}_d)$ is also real. Obviously, $(\tilde{\mathcal{E}}_c, \tilde{\mathcal{A}}_c) = \mathbf{t}(\mathcal{E}_d, \mathcal{A}_d)$ is real too. If only real symmetric ARE solutions are considered, the real version of all the results in this section can be derived in a similar way.

5 Transforming a CARE to a DARE

In this section we consider the relation between the CARE (2) and the DARE (1) under the transformation $\mathbf{f}_W^{-1}$. All results giving in this section are parallel to those in the last section. We will skip the proofs, since they are also similar to those in the last section.

Theorem 19 *Consider the CARE (2) and the DARE (1), where* $\mathbf{R}_d(X) = \mathbf{f}_W^{-1}(\mathbf{R}_c(X))$ *and the coefficient matrices of* $\mathbf{R}_d(X)$ *are obtained from (20) and (21). Suppose* $X \in \mathbb{H}^{n,n}$ *solves the CARE (2).*

(a) *If* $\det(R_d + B_d^*XB_d) \neq 0$*, then* X *also solves the DARE (1).*

(b) *If* $\det(R_d + B_d^*XB_d) = 0$*, then* X *cannot be a solution of the DARE.*

Theorem 20 *Consider the CARE (2) and the DARE (1), where* $\mathbf{R}_d(X) = \mathbf{f}_W^{-1}(\mathbf{R}_c(X))$*. Let* $(\mathcal{E}_d, \mathcal{A}_d)$ *be of the form (5). Suppose* $X \in \mathbb{H}^{n,n}$ *is a solution of the CARE (2) and* K_c *is of the form (14). If* $\det(R_d + B_d^*XB_d) \neq 0$*, then the matrices*

$$U_d = \begin{bmatrix} XE_d \\ I_n \\ K_d \end{bmatrix}, \quad Y_d = \begin{bmatrix} I_n \\ -A_d^*X \\ -B_d^*X \end{bmatrix}$$

satisfy

$$\mathcal{E}_d U_d = Y_d E_d, \quad \mathcal{A}_d U_d = Y_d(A_d + B_d K_d),$$

and the matrices

$$V_d = \begin{bmatrix} X(A_d + B_d K_d) \\ I_n \\ K_d \end{bmatrix}, \qquad Z_d = \begin{bmatrix} I_n \\ -E_d^* X \\ 0 \end{bmatrix}$$

satisfy

$$V_d^* \mathcal{E}_d = -(A_d + B_d K_d)^* Z_d^*, \quad V_d^* \mathcal{A}_d = -E_d^* Z_d^*,$$

where K_d is of the form (10).

Let

$$\hat{S} = W^{-1} \begin{bmatrix} I_n \\ K_c \end{bmatrix} =: \begin{bmatrix} \hat{S}_1 \\ \hat{S}_2 \end{bmatrix}.$$

The relations between K_c and K_d, $(E_c, A_c + B_c K_c)$ and $(E_d, A_d + B_d K_d)$ are given, respectively, by

$$\begin{bmatrix} I_n \\ K_d \end{bmatrix} = W^{-1} \begin{bmatrix} I_n \\ K_c \end{bmatrix} \hat{S}_1^{-1},$$

and

$$(E_d, A_d + B_d K_d) = \mathbf{c}^{-1} \left(\sqrt{2} E_c \hat{S}_1^{-1}, \sqrt{2}(A_c + B_c K_c) \hat{S}_1^{-1} \right).$$

Moreover, if X is a stabilizing (resp. semi-stabilizing) solution of the CARE (2), then X is also a stabilizing (resp. semi-stabilizing) solution of the DARE (1).

Although Theorem 19 and Theorem 13, Theorem 20 and Theorem 14 are similar, there is also a big difference: in Theorem 13, 14, the existence of a CARE is independent of the DARE solutions, but in Theorem 19, 20 the existence of a DARE does depend on the CARE solutions (more specifically, $\det(R_d + B_d^* X B_d)$). More details about the difference will be discussed below.

Theorem 21 *Consider the CARE (2) and the DARE (1), where $\mathbf{R}_d(X) = \mathbf{f}_W^{-1}(\mathbf{R}_c(X))$. Suppose that $X \in \mathbb{H}^{n,n}$ solves the CARE (2) and K_c is of the form (14).*

(a) *If $\det(A_c + B_c K_c - E_c) \neq 0$, then $\det(R_d + B_d^* X B_d) \neq 0$ for any W satisfying (20).*

(b) *If* $\det(A_c + B_cK_c - E_c) = 0$ *and* (E_c, A_c, B_c) *is controllable at* 1*, then* $\det(R_d + B_d^* X B_d) = 0$ *for all* W *satisfying (20).*

Theorem 22 *Consider the matrix pair* $(\mathcal{E}_c, \mathcal{A}_c)$ *of the form (6). Let* L_c *and* T_c *be basis matrices of* $\mathrm{null}[A_c - E_c,\ B_c]$ *and* $\mathrm{null}[A_c + E_c,\ B_c]$*, respectively. Then* $\det(\mathcal{A}_c - \mathcal{E}_c) \neq 0$ *if and only if* (E_c, A_c, B_c) *is controllable at both* -1 *and* 1 *and* $T_c^* \left[\begin{smallmatrix} M_c & N_c \\ N_c^* & R_c \end{smallmatrix}\right] L_c$ *is nonsingular.*

Remark 2 When $\det(A_c + B_cK_c - E_c) \neq 0$, we have $1 \notin \Lambda(E_c, A_c + B_cK_c)$ and $(E_c, A_c + B_cK_c)$ is regular. By (15), the pair $(\mathcal{E}_c, \mathcal{A}_c)$ is also regular. However, it does not mean that $\det(\mathcal{A}_c - \mathcal{E}_c) \neq 0$, since $1 \in \Lambda(-E_c^*, A_c + B_cK_c)$ is still possible. Certainly, when $\det(\mathcal{A}_c - \mathcal{E}_c) \neq 0$, then $\det(A_c + B_cK_c - E_c) \neq 0$ for any Hermitian solution X to the CARE (2).

When $\det(A_c + B_cK_c - E_c) = 0$, we have $1 \in \Lambda(E_c, A_c + B_cK_c)$ and/or $(E_c, A_c + B_cK_c)$ is singular. This also implies that $1 \in \Lambda(\mathcal{E}_c, \mathcal{A}_c)$ and/or $(\mathcal{E}_c, \mathcal{A}_c)$ is singular. Theorem 22 shows that this happens when at least one of the following holds, (a) (E_c, A_c, B_c) is not controllable at 1, (b) (E_c, A_c, B_c) is not controllable at -1, (c) $T_c^* \left[\begin{smallmatrix} M_c & N_c \\ N_c^* & R_c \end{smallmatrix}\right] L_c$ is singular.

When (E_c, A_c, B_c) is not controllable at 1, by Proposition 8 (ii), there are unitary matrices P_c, Q_c such that

$$E_c = P_c \begin{bmatrix} \hat{E}_{11} & \hat{E}_{12} \\ 0 & \hat{E}_{22} \end{bmatrix} Q_c,\ A_c = P_c \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} Q_c,\ B_c = P_c \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix}, \quad (38)$$

where $(\hat{E}_{11}, \hat{A}_{11}, \hat{B}_1)$ is controllable at 1 and $\Lambda(\hat{E}_{22}, \hat{A}_{22}) \subseteq \{1\}$. Partition

$$Q_c M_c Q_c^* = \begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} \\ \hat{M}_{12}^* & \hat{M}_{22} \end{bmatrix}, \qquad Q_c N_c = \begin{bmatrix} \hat{N}_1 \\ \hat{N}_2 \end{bmatrix},$$
$$P_c^* X P_c = \begin{bmatrix} \hat{X}_{11} & \hat{X}_{12} \\ \hat{X}_{12}^* & \hat{X}_{22} \end{bmatrix}, \qquad K_c Q_c^* = [\hat{K}_1, \hat{K}_2] \quad (39)$$

conformably, where $X \in \mathbb{H}^{n,n}$ and K_c is defined in (14). If X solves the CARE (2), then $\hat{X}_{11}$ solves the reduced CARE associated with the coefficient matrices $(\hat{E}_{11}, \hat{A}_{11}, \hat{B}_1, \hat{M}_{11}, \hat{N}_1, R_c)$, and $\hat{K}_1$ defined in (39) has the expression $\hat{K}_1 = -R_c^{-1}(\hat{B}_1^* \hat{X}_{11} \hat{E}_{11} + \hat{N}_1^*)$. Let $\hat{T} = [\hat{T}_{ij}]$ be nonsingular and satisfy

$$\frac{\sqrt{2}}{2} \begin{bmatrix} \hat{E}_{11} - \hat{A}_{11} & -\hat{B}_1 \\ \hat{E}_{11} + \hat{A}_{11} & \hat{B}_1 \end{bmatrix} \hat{T} = \begin{bmatrix} E_{11} & 0 \\ A_{11} & B_1 \end{bmatrix},$$

and define

$$W_c = \begin{bmatrix} Q_c^* & 0 \\ 0 & I_p \end{bmatrix} \left[\begin{array}{cc|c} \hat{T}_{11} & 0 & \hat{T}_{12} \\ 0 & I & 0 \\ \hline \hat{T}_{21} & 0 & \hat{T}_{22} \end{array}\right]. \tag{40}$$

Then W_c is nonsingular and satisfies (20).

Theorem 23 *Consider the CARE (2). Suppose that (A_c, B_c, E_c) is not controllable at 1 and it has the condensed form (38). Consider the DARE*

$$\mathbf{R}_d(X) = \mathbf{f}_{W_c}^{-1}(\mathbf{R}_c(X)) = 0,$$

where W_c is defined in (40). Suppose that $X \in \mathbb{H}^{n,n}$ is a solution of the CARE and $\hat{X}_{11}, \hat{K}_1$ are given in (39). If $\det(\hat{A}_{11} + \hat{B}_1\hat{K}_1 - \hat{E}_{11}) \neq 0$, then $\det(R_d + B_d^* X B_d) \neq 0$, where R_d, B_d are determined by (21) and (20), respectively, with $W = W_c$.

Proof. The proof is analogous to that of Theorem 17. □

Theorem 24 *Consider the CARE (2). Let $(\mathcal{E}_c, \mathcal{A}_c)$ be of the form (6) and $(\tilde{\mathcal{E}}_d, \tilde{\mathcal{A}}_d) = \mathbf{t}^{-1}(\mathcal{E}_c, \mathcal{A}_c)$, which is defined in (24). Suppose that $X \in \mathbb{H}^{n,n}$ and K_c is of the form (14). Then the following statements are equivalent.*

(i) *X solves the CARE.*

(ii) $\det R_c \neq 0$ *and the matrices*

$$\tilde{U}_d = \begin{bmatrix} X(E_c - A_c - B_c K_c) \\ I_n \\ K_c \end{bmatrix}, \quad \tilde{Y}_d = \begin{bmatrix} I_n \\ -(E_c + A_c)^* X \\ -B_c^* X \end{bmatrix}$$

satisfy

$$\tilde{\mathcal{E}}_d \tilde{U}_d = \frac{1}{2}\tilde{Y}_d(E_c - A_c - B_c K_c), \quad \tilde{\mathcal{A}}_d \tilde{U}_d = \frac{1}{2}\tilde{Y}_d(E_c + A_c + B_c K_c).$$

(iii) $\det R_c \neq 0$ *and the matrices*

$$\tilde{V}_d = \begin{bmatrix} X(E_c + A_c + B_c K_c) \\ I_n \\ K_c \end{bmatrix}, \quad \tilde{Z}_d = \begin{bmatrix} I_n \\ (A_c - E_c)^* X \\ B_c^* X \end{bmatrix}$$

satisfy

$$\tilde{V}_d^* \tilde{\mathcal{E}}_d = -\frac{1}{2}(E_c + A_c + B_c K_c)^* \tilde{Z}_d^*, \quad \tilde{V}_d^* \tilde{\mathcal{A}}_d = -\frac{1}{2}(E_c - A_c - B_c K_c)^* \tilde{Z}_d^*.$$

We also have the following properties. Let U_c and Y_c be defined in (13). Then

$$\tilde{U}_d + \tilde{V}_d = 2U_c, \qquad \tilde{Y}_d + \tilde{Z}_d = 2Y_c.$$

The matrix pairs associated with $\tilde{U}_d$ and U_c satisfy

$$\left(\frac{1}{2}(E_c - A_c - B_cK_c),\ \frac{1}{2}(E_c + A_c + B_cK_c)\right) = \mathbf{c}^{-1}(E_c,\ A_c + B_cK_c).$$

Theorem 24 may also be considered as a generalization of Theorem 20.

Finally, when $(\mathcal{E}_c, \mathcal{A}_c)$ is a real pair, if the matrix W is chosen to be real, the pair $(\mathcal{E}_d, \mathcal{A}_d) = \mathbf{f}_W^{-1}(\mathcal{E}_c, \mathcal{A}_c)$ is also real. Obviously, $(\tilde{\mathcal{E}}_d, \tilde{\mathcal{A}}_d) = \mathbf{t}^{-1}(\mathcal{E}_c, \mathcal{A}_c)$ is real too. If only real symmetric ARE solutions are considered, the real version of all the results in this section can be derived in a similar way.

6 Generalized transformations

Define the nonsingular matrix

$$\mathcal{X}_{\alpha,h} = \frac{1}{\sqrt{2h}} \begin{bmatrix} \alpha I_n & I_n \\ -\alpha h I_n & h I_n \end{bmatrix},$$

where $\alpha \in \mathbb{O}^0$ and $h > 0$. Given $(E_d, A_d, B_d, M_d, N_d, R_d)$, let W be nonsingular such that $\begin{bmatrix} A_d + \alpha E_d & B_d \end{bmatrix} W^{-1} = \begin{bmatrix} H & 0 \end{bmatrix}$, where $H \in \mathbb{C}^{n,n}$. Then we define the parameterized transformation

$$\mathbf{f}_{\alpha,h,W} : (E_d, A_d, B_d, M_d, N_d, R_d) \mapsto (E_c, A_c, B_c, M_c, N_c, R_c),$$

where $E_c, A_c, B_c, M_c, N_c, R_c$ satisfy

$$\begin{bmatrix} E_c & 0 \\ A_c & B_c \end{bmatrix} = \mathcal{X}_{\alpha,h} \begin{bmatrix} E_d & 0 \\ A_d & B_d \end{bmatrix} W^{-1}, \tag{41}$$

$$\begin{bmatrix} M_c & N_c \\ N_c^* & R_c \end{bmatrix} = W^{-*} \begin{bmatrix} M_d & N_d \\ N_d^* & R_d \end{bmatrix} W^{-1}. \tag{42}$$

Similarly, we define the corresponding transformation from $(E_c, A_c, B_c, M_c, N_c, R_c)$ to $(E_d, A_d, B_d, M_d, N_d, R_d)$, denoted by $\mathbf{f}_{\alpha,h,W}^{-1}$ (just as $\mathbf{f}_W^{-1}$ for $\tilde{\mathbf{f}}_{\tilde{W}}$ in Section 3), by

$$\begin{bmatrix} E_d & 0 \\ A_d & B_d \end{bmatrix} = \mathcal{X}_{\alpha,h}^{-1} \begin{bmatrix} E_c & 0 \\ A_c & B_c \end{bmatrix} W, \tag{43}$$

$$\begin{bmatrix} M_d & N_d \\ N_d^* & R_d \end{bmatrix} = W^* \begin{bmatrix} M_c & N_c \\ N_c^* & R_c \end{bmatrix} W. \tag{44}$$

Note that $\mathcal{X}_{1,1} = \mathcal{X}$ and $\mathbf{f}_{1,1,W} = \mathbf{f}_W$. So the transformation $\mathbf{f}_{\alpha,h,W}$ is a generalization of $\mathbf{f}_W$. Clearly, the transformation $\mathbf{f}_{\alpha,h,W}$ also relates the DARE (1) and CARE (2), and their associated matrix pairs $(\mathcal{E}_d, \mathcal{A}_d)$ and $(\mathcal{E}_c, \mathcal{A}_c)$.

Just as $\mathbf{f}_W$ has the relation with $\mathbf{c}$, the transformation $\mathbf{f}_{\alpha,h,W}$ has the relation with the generalized Cayley transformation $\mathbf{c}_{\alpha,h}$ defined by

$$\mu = \mathbf{c}_{\alpha,h}(\lambda) = h(\lambda - \alpha)(\lambda + \alpha)^{-1} = h(\bar{\alpha}\lambda - 1)(\bar{\alpha}\lambda + 1)^{-1} = h\mathbf{c}(\bar{\alpha}\lambda).$$

(Note $\mathbf{c} = \mathbf{c}_{1,1}$.) The relation between λ and μ is summarized in Table 2.

λ	$\lvert\lambda\rvert < 1$	$\lvert\lambda\rvert = 1$	$\lvert\lambda\rvert > 1$	α	0	$-\alpha$	∞
μ	$\operatorname{Re}\mu < 0$	$\operatorname{Re}\mu = 0$	$\operatorname{Re}\mu > 0$	0	$-h$	∞	h

Table 2: Correspondence between λ and $\mu = \mathbf{c}_{\alpha,h}(\lambda)$

The transformation $\mathbf{c}_{\alpha,h}$ can also be applied to matrix pairs:

$$\mathbf{c}_{\alpha,h}(\mathcal{E}, \mathcal{A}) = (\mathcal{A} + \alpha\mathcal{E}, h(\mathcal{A} - \alpha\mathcal{E}))\,.$$

For fixed α and h, the transformation $\mathbf{f}_{\alpha,h,W}$ behaves completely the same as $\mathbf{f}_W$. In the previous three sections we have seen that -1 and 1, the poles of $\mathbf{c}$ and $\mathbf{c}^{-1}$, respectively, are responsible for the possible failure of $\mathbf{f}_W$ and $\mathbf{f}_W^{-1}$. Similarly, the poles of $\mathbf{c}_{\alpha,h}$ and $\mathbf{c}_{\alpha,h}^{-1}$, which are $-\alpha$ and h, respectively, may cause the same problems to $\mathbf{f}_{\alpha,h,W}$ and $\mathbf{f}_{\alpha,h,W}^{-1}$. The same results as in Sections 3 – 5 can be derived for $\mathbf{f}_{\alpha,h,W}$ and $\mathbf{f}_{\alpha,h,W}^{-1}$. The only change is that the poles -1 and 1 are replaced by $-\alpha$ and h, respectively, wherever they appear.

However, the parameters in $\mathbf{f}_{\alpha,h,W}$ give it some advantages. For a given ARE, we now are able to select α and h so that the problems happened to $\mathbf{f}_W$ may possibly be avoided.

Theorem 25 *Consider the DARE (1). Let $(\mathcal{E}_d, \mathcal{A}_d)$ be the associated matrix pair of the form (5).*

(a) *If $(\mathcal{E}_d, \mathcal{A}_d)$ is regular, then one can always select an $\alpha_0 \in \mathbb{O}^0$ such that $\det(\mathcal{A}_d + \alpha_0\mathcal{E}_d) \neq 0$. Let $\mathbf{f}_{\alpha_0,h,W}$ be a transformation with arbitrary $h > 0$ and W satisfying (41). Then any solution $X \in \mathbb{H}^{n,n}$ of the DARE (1) also solves the CARE*

$$\mathbf{R}_c(X) = \mathbf{f}_{\alpha_0,h,W}(\mathbf{R}_d(X)) = 0.$$

(b) *If* $(\mathcal{E}_d, \mathcal{A}_d)$ *is singular and* (E_d, A_d, B_d) *is controllable in* $\mathbb{O}^0$*, then for any* $\mathbf{f}_{\alpha,h,W}$ *with* $\alpha \in \mathbb{O}^0$ *and* $h > 0$*,* R_c *defined in (42) is singular and there is no corresponding CARE.*

Proof. In part (a) the existence of α_0 is based on the regularity of $(\mathcal{E}_d, \mathcal{A}_d)$. The rest of the proof is the same as that of Theorem 13. □

When $(\mathcal{E}_d, \mathcal{A}_d)$ is singular and (E_d, A_d, B_d) is not controllable at some numbers in $\mathbb{O}^0$, one may consider the reduced DARE by using the decomposition (4), as did in Theorem 17.

Theorem 26 *Consider the CARE (2). Let* $(\mathcal{E}_c, \mathcal{A}_c)$ *be the associated matrix pair of the form (6).*

(a) *If* $(\mathcal{E}_c, \mathcal{A}_c)$ *is regular, then one can always select an* $h_0 > 0$ *such that* $\det(\mathcal{A}_c - h_0\mathcal{E}_c) \neq 0$*. Let* $\mathbf{f}_{\alpha,h_0,W}$ *be a transformation with arbitrary* $\alpha \in \mathbb{O}^0$ *and* W *satisfying (43). Then any solution* $X \in \mathbb{H}^{n,n}$ *of the CARE (2) also solves the DARE*

$$\mathbf{R}_d(X) = \mathbf{f}^{-1}_{\alpha,h_0,W}(\mathbf{R}_c(X)) = 0. \tag{45}$$

(b) *If* $(\mathcal{E}_c, \mathcal{A}_c)$ *is singular and* (E_c, A_c, B_c) *is controllable in* $\{h|h > 0\}$*, then for any* $\mathbf{f}^{-1}_{\alpha,h,W}$ *with* $\alpha \in \mathbb{O}^0$ *and* $h > 0$*, the matrix* $R_d + B_d^* X B_d$ *with* R_d, B_d *defined by (44) and (43) is singular for any solution* $X \in \mathbb{H}^{n,n}$ *of the CARE, and no Hermitian solution of the CARE solves the DARE (45).*

Proof. In part (a) the existence of h_0 is based on the regularity of $(\mathcal{E}_c, \mathcal{A}_c)$. The rest of the proof is the similar to that of Theorem 13. □

Similarly, when $(\mathcal{E}_c, \mathcal{A}_c)$ is singular and (E_c, A_c, B_c) is not controllable at some $h > 0$, one may consider the reduced CARE by using the decomposition (4), as did in Theorem 23.

There is also an intermediate transformation $\mathbf{t}_{\alpha,h}$ of $\mathbf{f}_{\alpha,h,W}$, which is a generalization of $\mathbf{t}$. We may apply $\mathbf{t}_{\alpha,h}$ and $\mathbf{t}^{-1}_{\alpha,h}$, respectively, to $(\mathcal{E}_d, \mathcal{A}_d)$ and $(\mathcal{E}_c, \mathcal{A}_c)$ to obtain

$$\left(\hat{\mathcal{E}}_c, \hat{\mathcal{A}}_c\right) = \mathbf{t}_{\alpha,h}(\mathcal{E}_d, \mathcal{A}_d) = \left(\begin{bmatrix} 0 & A_d + \alpha E_d & B_d \\ -(A_d + \alpha E_d)^* & 0 & 0 \\ -B_d^* & 0 & 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 0 & h(A_d - \alpha E_d) & hB_d \\ h(A_d - \alpha E_d)^* & M_d & N_d \\ hB_d^* & N_d^* & R_d \end{bmatrix}\right),$$

$$\left(\hat{\mathcal{E}}_d, \hat{\mathcal{A}}_d\right) = \mathbf{t}_{\alpha,h}^{-1}(\mathcal{E}_c\, \mathcal{A}_c) = \left(\begin{bmatrix} 0 & \frac{\bar{\alpha}}{2h}(hE_c - A_c) & -\frac{\bar{\alpha}}{2h}B_c \\ -\frac{1}{2h}(hE_c + A_c)^* & 0 & 0 \\ -\frac{1}{2h}B_c^* & 0 & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 0 & \frac{1}{2h}(hE_c + A_c) & \frac{1}{2h}B_c \\ -\frac{\alpha}{2h}(hE_c - A_c)^* & M_c & N_c \\ \frac{\alpha}{2h}B_c^* & N_c^* & R_c \end{bmatrix} \right).$$

The following theorems are parallel to Theorem 18 and Theorem 24, respectively.

Theorem 27 *Consider the DARE (1) and the associated pair* $(\mathcal{E}_d, \mathcal{A}_d)$. *Let* $(\hat{\mathcal{E}}_c, \hat{\mathcal{A}}_c) = \mathbf{t}_{\alpha,h}(\mathcal{E}_d, \mathcal{A}_d)$. *Then* $X \in \mathbb{H}^{n,n}$ *solves the DARE (1) if and only if* $\det(R_d + B_d^* X B_d) \neq 0$ *and the matrices*

$$\hat{U}_c = \begin{bmatrix} X(A_d + B_d K_d + \alpha E_d) \\ 2hI_n \\ 2hK_d \end{bmatrix}, \quad \hat{Y}_c = \begin{bmatrix} 2hI_n \\ -(A_d + \alpha E_d)^* X \\ -B_d^* X \end{bmatrix}$$

satisfy

$$\hat{\mathcal{E}}_c \hat{U}_c = \hat{Y}_c(A_d + B_d K_d + \alpha E_d), \quad \hat{\mathcal{A}}_c \hat{U}_c = \hat{Y}_c(A_d + B_d K_d - \alpha E_d)$$

as well as

$$\hat{U}_c^* \hat{\mathcal{E}}_c = -(A_d + B_d K_d + \alpha E_d)^* \hat{Y}_c^*, \quad \hat{U}_c^* \hat{\mathcal{A}}_c = (A_d + B_d K_d - \alpha E_d)^* \hat{Y}_c^*,$$

where K_d *is defined in (10).*

Proof. The proof is analogous to that of Theorem 18. □

Theorem 28 *Consider the CARE (2) and the associated pair* $(\mathcal{E}_c, \mathcal{A}_c)$. *Let* $(\hat{\mathcal{E}}_d, \hat{\mathcal{A}}_d) = \mathbf{t}_{\alpha,h}^{-1}(\mathcal{E}_c, \mathcal{A}_c)$. *Suppose that* $X \in \mathbb{H}^{n,n}$ *and* K_c *is of the form (14). Then the following statements are equivalent.*

(i) X *solves the CARE (2).*

(ii) $\det R_c \neq 0$ *and the matrices*

$$\hat{U}_d = \begin{bmatrix} \bar{\alpha} X(hE_c - (A_c + B_c K_c)) \\ I_n \\ K_c \end{bmatrix}, \quad \hat{Y}_d = \begin{bmatrix} I_n \\ -(hE_c + A_c)^* X \\ -B_c^* X \end{bmatrix}$$

satisfy

$$\begin{aligned}\hat{\mathcal{E}}_d\hat{U}_d &= \hat{Y}_d\left(\frac{\bar{\alpha}}{2h}(hE_c-(A_c+B_cK_c))\right),\\ \hat{\mathcal{A}}_d\hat{U}_d &= \hat{Y}_d\left(\frac{1}{2h}(hE_c+(A_c+B_cK_c))\right).\end{aligned}$$

(iii) $\det R_c \neq 0$ *and the matrices*

$$\hat{V}_d = \begin{bmatrix} X(hE_c+(A_c+B_cK_c)) \\ I_n \\ K_c \end{bmatrix}, \quad \hat{Z}_d = \begin{bmatrix} I_n \\ \alpha(A_c-hE_c)^*X \\ \alpha B_c^*X \end{bmatrix}$$

satisfy

$$\begin{aligned}\hat{V}_d^*\hat{\mathcal{E}}_d &= \left(-\frac{1}{2h}(hE_c+(A_c+B_cK_c))^*\right)\hat{Z}_d^*,\\ \hat{V}_d^*\hat{\mathcal{A}}_d &= \left(-\frac{\alpha}{2h}(hE_c-(A_c+B_cK_c))^*\right)\hat{Z}_d^*.\end{aligned}$$

Proof. The proof is analogous to that of Theorem 18. □

Remark 3 The transformation given in [11] is a special case of $\mathbf{f}_{\alpha,h,W}$ with $h=1$ and

$$W = \begin{bmatrix} \sqrt{2}(A+\alpha E)^{-1} & -(A+\alpha E)^{-1}B \\ 0 & I \end{bmatrix}.$$

(In [11], $E=I$.) For this transformation, α needs to be selected such that $A+\alpha E$ is invertible, and the inverse $(A+\alpha E)^{-1}$ presents explicitly in W.

We finally mention that $\mathbf{f}_{\alpha,h,W}$ also has some limitations. When the DARE (resp. CARE) is real, i.e., all its coefficient matrices are real, it is natural to require the corresponding CARE (resp. DARE) to be real. Then among all the transformations $\mathbf{f}_{\alpha,h,W}$, one can only use either $\mathbf{f}_{1,1,W} = \mathbf{f}_W$ or $\mathbf{f}_{-1,1,W}$.

7 Conclusion

We have introduced the transformations $\mathbf{f}_W$ and $\mathbf{f}_{\alpha,h,W}$ that simply relate the discrete-time and continuous-time algebraic Riccati equations. The transformations make it possible to study and solve the two types of AREs and their associated control problems in a unified way. For the discrete-time and continuous-times linear quadratic optimal control problems, the transformations connect not only the associated AREs but also the control problems themselves. The AREs from robust control are usually more complicated. In order to use the transformations to connect the discrete-time and continuous-time robust control problems, further work needs to done. Both transformations may fail for some AREs. Further study is also needed to deal with this problem.

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