A robust numerical method for the γ-iteration in H_∞ control

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Abstract

We present a numerical method for the solution of the optimal H_∞ control problem based on the γ-iteration and a novel extended matrix pencil formulation of the state-space solution to the (sub)optimal H_∞ control problem. In particular, instead of algebraic Riccati equations or unstructured matrix pencils, our approach is solely based on solving even generalized eigenproblems. The enhanced numerical robustness of the method is derived from the fact that using the structure of the problem, spectral symmetries are preserved. Moreover, these methods are also applicable even if the pencil has eigenvalues on the imaginary axis. We compare the new method with conventional methods and present several examples.

Keywords. H_∞ control, algebraic Riccati equation, CS decomposition, Lagrangian subspaces, even matrix pencil
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1 Introduction

The optimal infinite-horizon output (or measurement) feedback H_∞ control problem is one of the central tasks in robust control, see, e.g., [28, 40, 50, 54]. Nevertheless, the development of robust numerical methods for the H_∞ control is unusually difficult [48]. It remains a major open problem [16] despite recent developments [19, 23, 26, 29, 41, 44] some of which are incorporated into software libraries like SLICOT[1] [8, 12, 30] or the MATLAB Robust Control Toolbox [2].

This paper derives a numerical method based on a better exploitation of symmetry structures in the underlying linear algebra problems. It is therefore expected that the resulting

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[1]See http://www.slicot.org

method will exhibit some better robustness in the presence of rounding errors than other methods.

If $F(s)$ is a matrix valued analytic function that is bounded in the open right-half plane, then its H_∞ norm is $\|F\|_\infty = \sup_{\omega\in\mathbb{R}} \sigma_{max}[F(\imath\omega)]$, where $\sigma_{max}[F(\imath\omega)]$ denotes the maximal singular value of the matrix $F(\imath\omega)$ and $\imath = \sqrt{-1}$. In robust control, the H_∞ norm of a transfer function from disturbance inputs to error outputs is a measure of the worst case influence of disturbances. The optimal H_∞ control problem is the task of designing a dynamic controller that minimizes this measure.

Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \qquad x(t_0) = x^0, \\ z &= C_1 x + D_{11} w + D_{12} u, \\ y &= C_2 x + D_{21} w + D_{22} u, \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n,n}$, $B_i \in \mathbb{R}^{n,m_i}$, $C_i \in \mathbb{R}^{p_i,n}$, and $D_{ij} \in \mathbb{R}^{p_i,m_j}$ for $i,j = 1,2$. (By $\mathbb{R}^{n,k}$ we denote the set of real $n \times k$ matrices.) As usual, see [28, 54], we assume $p_1 \geq m_2$ and $m_1 \geq p_2$. In this system, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{m_2}$ is the control input vector, and $w(t) \in \mathbb{R}^{m_1}$ is an exogenous input that may include noise, linearization errors and unmodeled dynamics. The vector $y(t) \in \mathbb{R}^{p_2}$ contains measured outputs, while $z(t) \in \mathbb{R}^{p_1}$ is a regulated output or an estimation error.

Definition 1.1 The Optimal H_∞ Control Problem: Determine a controller (dynamic compensator)

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}y, \\ u &= \hat{C}\hat{x} + \hat{D}y, \end{aligned} \tag{2}$$

with $\hat{A} \in \mathbb{R}^{N,N}$, $\hat{B} \in \mathbb{R}^{N,p_2}$, $\hat{C} \in \mathbb{R}^{m_2,N}$, $\hat{D} \in \mathbb{R}^{m_2,p_2}$ and transfer function $K(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D}$ such that the closed-loop system resulting from (1) and (2),

$$\begin{aligned} \dot{x} &= (A + B_2\hat{D}Z_1C_2)x + (B_2Z_2\hat{C})\hat{x} + (B_1 + B_2\hat{D}Z_1D_{21})w, \\ \dot{\hat{x}} &= \hat{B}Z_1C_2x + (\hat{A} + \hat{B}Z_1D_{22}\hat{C})\hat{x} + \hat{B}Z_1D_{21}w, \\ z &= (C_1 + D_{12}Z_2\hat{D}C_2)x + D_{12}Z_2\hat{C}\hat{x} + (D_{11} + D_{12}\hat{D}Z_1D_{21})w, \end{aligned} \tag{3}$$

with $Z_1 = (I - D_{22}\hat{D})^{-1}$ and $Z_2 = (I - \hat{D}D_{22})^{-1}$, is *internally stable*, (i.e., for $w \equiv 0$ the closed-loop system is asymptotically stable), and the closed-loop transfer function $T_{zw}(s)$ from w to z is minimized in the H_∞ norm.

The solution of the problem is, in general, difficult. Solving the H_∞ control problem by directly minimizing $\|T_{zw}\|_\infty$ over the set of internally stabilizing controllers (2) is intractable by conventional optimization methods. It is often unclear whether a minimizing controller exists [54, p.414]. When a minimizing controller or an approximately minimizing controller does exist, it is typically not unique. Nevertheless, the well-known state-space solution to the H_∞ control problem [21, 22], relating H_∞ control to algebraic Riccati equations, provides a way to solve many H_∞ control problems despite the above difficulties. We review this solution in Subsection 2.2 following [54]. In summary, for each number $\gamma > 0$, the theory is based on an explicit computational test for the existence of an internally stabilizing dynamic controller (3) whose closed-loop transfer function $T_{zw}(s)$ satisfies $\|T_{zw}\|_\infty < \gamma$. Explicit but complicated formulas in terms of γ for a dynamic controller that achieves $\gamma > \|T_{zw}\|_\infty$ (when

one exists) appear, e.g., in [29, 54] and are discussed in [7]. Hence, at least in principle, the H_∞ control problem can be solved by bisection (or alike) on γ.

Here, we divide the optimal H_∞ problem into two subproblems that we call the *modified optimal H_∞ control problem* and the *suboptimal H_∞ control problem*.

Definition 1.2 The Modified Optimal H_∞ Control Problem: Let Γ be the set of numbers $\gamma > 0$ for which there exists an internally stabilizing dynamic controller with transfer function $T_{zw}(s)$ satisfying $\gamma > \|T_{zw}\|_\infty$. Determine $\gamma_{mo} = \inf \Gamma$. (If no internally stabilizing controller exists, then $\Gamma = \emptyset$ and $\gamma_{mo} = \infty$.)

The modified optimal H_∞ control problem is an optimization in the single independent variable γ, while the optimal H_∞ control problem requires optimization over the complicated set of stabilizing controllers.

Because in many applications, it is neither practical nor necessary to determine γ_{mo} to high precision and because there may be no dynamic controller so that γ_{mo} is actually attained, in general, it is necessary to use a controller whose transfer function has larger H_∞ norm, i.e., a *suboptimal controller*.

Definition 1.3 The Suboptimal H_∞ Control Problem: For a given value $\gamma \in \Gamma$, find an internally stabilizing dynamic controller such that the closed loop transfer function satisfies $\|T_{zw}\|_\infty < \gamma$.

The process of solving the modified optimal H_∞ control problem is sometimes called the *γ-iteration*. Once a sufficiently accurate approximation to γ_{mo} has been determined, a suboptimal controller may be constructed using the formulas suggested in [29, 54] or by the more robust formulas in [7]. In this paper we present a rounding-error robust numerical method for the γ-iteration.

The outline of the paper is as follows. First, we introduce some necessary notation and review some of the theory surrounding H_∞ control in Section 2. In Section 3 we discuss some of the existing numerical methods and point out where numerical difficulties may arise. In Section 4 we present a formulation of the modified optimal H_∞ control problem chosen to avoid such numerical difficulties. The formulation incorporates ideas from singular H_∞ control [19, 26] in combination with numerical methods designed especially for even eigenvalue problems [5, 15]. Using structure preserving methods for these eigenvalue problem, we derive a numerically robust γ-iteration in Section 5. The procedure applies in situations where classical γ-iterations fail. Consequently, it allows the H_∞ approach to be used on a broader range of problems.

2 Preliminaries and Theoretical Background

2.1 Hamiltonian matrices and algebraic Riccati equations

In this section we introduce some notation and definitions. By $\mathbb{R}^{n,k}$ we denote the set of real $n \times k$ matrices and I_n is the identity matrix in $\mathbb{R}^{n,n}$. For symmetric matrices A and B, $A \geq B$ and $A > B$ mean that $A - B$ is positive semidefinite and positive definite, respectively. An eigenvalue λ of a square matrix A is *stable (semi-stable)* if its real part is negative (zero). A square matrix A is *stable* (*semi-stable*) if all the eigenvalues of A are in the open (closed) left half complex plane.

Definition 2.1 *Let* $\mathcal{J} := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$.

a) $\mathcal{H} \in \mathbb{R}^{2n,2n}$ *is* Hamiltonian *if* $(\mathcal{H}\mathcal{J})^T = \mathcal{H}\mathcal{J}$ *and it is* skew-Hamiltonian *if* $(\mathcal{H}\mathcal{J})^T = -\mathcal{H}\mathcal{J}$.

b) $\mathcal{Z} \in \mathbb{R}^{2n,2n}$ *is* symplectic *if* $\mathcal{Z}\mathcal{J}\mathcal{Z}^T = \mathcal{J}$, *and* $\mathcal{U} \in \mathbb{R}^{2n,2n}$ *is* orthogonal symplectic *if* $\mathcal{U}\mathcal{J}\mathcal{U}^T = \mathcal{J}$ *and* $\mathcal{U}^T\mathcal{U} = I_{2n}$.

c) An invariant subspace $\mathcal{L}$ *of a Hamiltonian matrix* $\mathcal{H} \in \mathbb{R}^{2n,2n}$ *is* Lagrangian *if it is* n*-dimensional and* $x^H J y = 0$, *for all* x, $y \in \mathcal{L}$.

d) An invariant subspace $\mathcal{L}$ *of a Hamiltonian matrix* $\mathcal{H} \in \mathbb{R}^{2n,2n}$ *is* stable (semi-stable) Lagrangian *if it is Lagrangian and corresponds to the stable (semi-stable) eigenvalues of* $\mathcal{H}$.

Real Hamiltonian matrices take the form

$$\mathcal{H} = \begin{bmatrix} F & -G \\ -K & -F^T \end{bmatrix}, \tag{4}$$

where F, G, $K \in \mathbb{R}^{n,n}$, $G = G^T$ and $K = K^T$.

An important property of real Hamiltonian matrices is their spectral symmetry: the eigenvalues are symmetric about both the real axis and the imaginary axis, see [33, 35, 37]. Eigenvalues with nonzero real and imaginary parts occur in quadruples consisting of two $\pm$ pairs, λ, $-\lambda$, $\bar{\lambda}$, $-\bar{\lambda}$. Real eigenvalues and pure imaginary eigenvalues appear in $\pm$ pairs.

To each Hamiltonian matrix there corresponds an *algebraic Riccati equation (ARE)*

$$F^T X + XF + K - XGX = 0. \tag{5}$$

Definition 2.2 A matrix X is a *stabilizing* (*semi-stabilizing*) solution of (5) if $X = X^T$ and $F - GX$ is stable (semi-stable).

It is well known [33, 37] and easy to verify that if X is a stabilizing (semi-stabilizing) solution of the ARE (5), then the columns of $\begin{bmatrix} I_n \\ X \end{bmatrix}$ span a stable (semi-stable) Lagrangian invariant subspace of the Hamiltonian matrix (4). Conversely, if the columns of $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ span a stable (semi-stable) Lagrangian invariant subspace of the Hamiltonian matrix (4) and if X_1 is non-singular, then $X = X_2 X_1^{-1}$ is a stabilizing (semi-stabilizing) solution of the ARE (5). But note that a (semi-)stable Lagrangian subspace of a Hamiltonian matrix may exist even if the ARE (5) does not have a positive semidefinite solution, see [7, 25].

Conventional numerical methods for the modified optimal H_∞ control problem require the computation of the stabilizing solution of AREs of the form (5) in which F and/or G are not necessarily semidefinite or for which (K, F) is not stabilizable or (K, G) is not detectable. Such AREs may have no positive semidefinite semi-stabilizing solution [33, 37]. This is one source of trouble in numerical methods for the modified optimal H_∞ control problem. This paper presents a numerical method that circumvents this problem by directly computing the semi-stabilizing Lagrangian subspace.

2.2 State-space solution of the optimal H_∞ control problem

In this section we discuss the theoretical background for the modified optimal H_∞ problem. We start with a typical set of assumptions [29, 28, 40, 54].

Assumptions:

A1. The pair (A, B_2) is *stabilizable* and the pair (A, C_2) is *detectable*.

A2. $D_{22} = 0$ and both D_{12} and D_{21} have full rank.

A3. The matrix $\left[\begin{smallmatrix} A-i\omega I & B_2 \\ C_1 & D_{12} \end{smallmatrix}\right]$ has full column rank for all real ω.

A4. The matrix $\left[\begin{smallmatrix} A-i\omega I & B_1 \\ C_2 & D_{21} \end{smallmatrix}\right]$ has full row rank for all real ω.

Remark 2.3 The requirement that $D_{22} = 0$ (Assumption A2) is for convenience. Systems that have a feedthrough term can be synthesized by first studying the problem without this term, see [54].

In the literature, it is often assumed that $D_{12} = \left[\begin{smallmatrix} 0 \\ I_{m_2} \end{smallmatrix}\right]$ and $D_{21} = [0\, I_{p_2}]$ and that $D_{11} = 0$. In principle, this particular form can be obtained from a more general system by transforming the system in advance as does MATLAB's `hinfsyn` [2]. Unfortunately, reducing the system to this form may require ill-conditioned transformations that lead to unnecessary numerical errors. Thus, we allow general D_{12}, D_{21} and D_{11} subject to Assumption A2. Note that this leads to slightly different solution formulas for the optimal feedbacks and the closed-loop system than those given in [29, 54], see [7].

To formulate the basic theorem of H_∞ control, we introduce the following two symmetric matrices depending on the D_{ij} and a parameter $\gamma \in \mathbb{R}$,

$$\begin{aligned} R_H(\gamma) &:= \begin{bmatrix} D_{11}^T \\ D_{12}^T \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \end{bmatrix} - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}, \\ R_J(\gamma) &:= \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} \begin{bmatrix} D_{11}^T & D_{21}^T \end{bmatrix} - \begin{bmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{6}$$

Conventional H_∞ numerical methods require that both $R_H(\gamma)$ and $R_J(\gamma)$ are nonsingular. The following proposition provides a convenient test.

Proposition 2.4 *[7, 54] If Assumption A2 is not satisfied, then either R_H is singular for all γ or R_J is singular for all γ.*

If Assumption A2 holds, then there exist only a finite number of values of $\gamma \geq 0$ for which one or both of the matrices $R_H(\gamma)$ or $R_J(\gamma)$ is singular.

Definition 2.5 Define $\hat{\gamma}_H$, $\hat{\gamma}_J$ and $\hat{\gamma}$ by $\hat{\gamma}_H := \max\{\gamma \in \mathbb{R} \mid R_H(\gamma) \text{ is singular}\}$, $\hat{\gamma}_J := \max\{\gamma \in \mathbb{R} \mid R_J(\gamma) \text{ is singular}\}$ and $\hat{\gamma} := \max\{\hat{\gamma}_H, \hat{\gamma}_J\}$.

If $D_{11} = 0$, then $\hat{\gamma} = 0$. If $D_{11} \neq 0$, then $\hat{\gamma}$ is typically positive. Let $D_{12} = U_{12} \left[\begin{smallmatrix} 0 \\ \Sigma_{12} \end{smallmatrix}\right] V_{12}^T$ and $D_{21} = V_{21}[0\, \Sigma_{21}]U_{21}^T$ be (slightly permuted) singular value decompositions of D_{12} and D_{21}

with real orthogonal matrices U_{12}, U_{21}, V_{12}, V_{21} and positive diagonal matrices Σ_{12} and Σ_{21}. Use the orthogonal equivalence transformation

$$\left[\begin{array}{c|c} U_{12}^T & 0 \\ \hline 0 & V_{21}^T \end{array}\right] \left[\begin{array}{c|c} D_{11} & D_{12} \\ \hline D_{21} & 0 \end{array}\right] \left[\begin{array}{c|c} U_{21} & 0 \\ \hline 0 & V_{12} \end{array}\right] = \left[\begin{array}{cc|c} D_1 & D_2 & 0 \\ D_3 & D_4 & \Sigma_{12} \\ \hline 0 & \Sigma_{21} & 0 \end{array}\right]$$

to define D_1, D_2, D_3 and D_4.

Proposition 2.6 *[54] If Assumption A2 holds, then* $\hat{\gamma}_H = \sigma_{max}[D_1\, D_2]$, $\hat{\gamma}_J = \sigma_{max}\left[\begin{smallmatrix} D_1 \\ D_3 \end{smallmatrix}\right]$ *and the following equivalences hold.*

i) $R_H(\gamma)$ *is invertible if and only if* $D_1 D_1^T + D_2 D_2^T - \gamma^2 I$ *is invertible.*

ii) $R_J(\gamma)$ *is invertible if and only if* $D_1^T D_1 + D_3^T D_3 - \gamma^2 I$ *is invertible.*

The next theorem gives the theoretical basis for the γ-iteration.

Theorem 2.7 *[54]. Consider system (1), with* R_H *and* R_J *as in (6). Under assumptions A1–A4, there exists an internally stabilizing controller such that the transfer function from w to z, denoted by* T_{zw}*, satisfies* $\|T_{zw}\|_\infty < \gamma$ *if and only if the following four conditions hold.*

1. $\gamma > \hat{\gamma}$ *with* $\hat{\gamma}$ *as in Definition 2.5.*

2. *There exists a stabilizing positive semidefinite solution* $X_H = X_H(\gamma)$ *of the ARE associated with the Hamiltonian matrix*

$$H(\gamma) = \begin{bmatrix} A & 0 \\ -C_1^T C_1 & -A^T \end{bmatrix} - \begin{bmatrix} B_1 & B_2 \\ -C_1^T D_{11} & -C_1^T D_{12} \end{bmatrix} R_H^{-1}(\gamma) \begin{bmatrix} D_{11}^T C_1 & B_1^T \\ D_{12}^T C_1 & B_2^T \end{bmatrix}, \quad (7)$$

3. *There exists a stabilizing positive semidefinite solution* $X_J = X_J(\gamma)$ *of the ARE associated with the Hamiltonian matrix*

$$J(\gamma) = \begin{bmatrix} A^T & 0 \\ -B_1 B_1^T & -A \end{bmatrix} - \begin{bmatrix} C_1^T & C_2^T \\ -B_1 D_{11}^T & -B_1 D_{21}^T \end{bmatrix} R_J^{-1}(\gamma) \begin{bmatrix} D_{11} B_1^T & C_1 \\ D_{21} B_1^T & C_2 \end{bmatrix}. \quad (8)$$

4. $\gamma^2 > \rho(X_H X_J)$. *(Here* $\rho(X_H X_J)$ *denotes the spectral radius of* $X_H X_J$*.)*

The solution to the suboptimal control problem, γ_{mo}, is the supremum of all $\gamma \geq 0$ for which at least one of the conditions in Theorem 2.7 fails.

3 Conventional Numerical Methods

This section discusses finite precision arithmetic hazards encountered by typical numerical methods for checking the four conditions in Theorem 2.7. Some finite precision hazards are also discussed in [26, 29, 48].

Conventional numerical methods for the solution of the modified optimal H_∞ problem [41, 48] fall into two categories.

The first embeds the problem into an optimization problem with two linear matrix inequality (LMI) constraints [13] and employs methods of semidefinite programming to find

γ_{mo}. This is attractive, because easy-to-use methods for semidefinite programming are available, see, e.g., [38]. In such an approach, LMIs in $\mathcal{O}(n^2)$ variables need to be solved which in general results in a complexity of $\mathcal{O}(n^6)$. Despite recent progress in reducing this complexity based on exploiting duality in the related semidefinite programs [1, 52], the best complexity achievable is still larger than $\mathcal{O}(n^4)$ as compared to the $\mathcal{O}(n^3)$ cost of the procedure discussed here.

The second is the category of *Riccati methods*. A typical Riccati method uses Theorem 2.7 to find upper and lower bounds on γ_{mo} which are then refined by bisection also using Theorem 2.7. A quadratically convergent algorithm based on Newton's method can be found in [45]. Each iterative step includes checking whether $\gamma > \hat{\gamma}$, using an ARE solver like those discussed in [3, 20, 37, 46] to compute stabilizing solutions X_H and X_J (if they exist) corresponding to (7) and (8), and then checking whether $\gamma^2 > \rho(X_H X_J)$. This method has complexity $\mathcal{O}(n^3)$ per step. Variations of this approach employ stable Lagrangian invariant subspaces of the Hamiltonian matrices associated with the AREs (7) and (8) [54] or deflating subspaces of corresponding matrix pencils [19, 44, 26]. At this writing, Riccati methods are the only practical choice for higher dimensional problems.

Unfortunately, there are several numerical difficulties associated with Riccati methods. Primary among these is the fact that often as γ approaches γ_{mo}, one of the ARE solutions X_H or X_J either diverges to ∞ or becomes highly ill-conditioned, i.e., tiny errors in the Hamiltonian matrices $H(\gamma)$ or $J(\gamma)$ may lead to large errors in X_H or X_J. The following example demonstrates this.

Example 3.1 Consider the system

$$\left[\begin{array}{c|c|c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{21} & \end{array}\right] = \left[\begin{array}{cc|cc|c} -1 & 0 & \epsilon_1 & 0 & 1 \\ 0 & -1 & 0 & \epsilon_2 & 1 \\ \hline \alpha & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \beta & 0 & \frac{1}{2} & 1 \\ \hline \delta & \eta & 0 & 1 & 0 \end{array}\right].$$

If $\epsilon_1 = \epsilon_2 = 0$, then (6) becomes

$$R_H(\gamma) = R_J(\gamma) = \left[\begin{array}{ccc} \frac{1}{4} - \gamma^2 & 0 & 0 \\ 0 & \frac{1}{4} - \gamma^2 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \end{array}\right],$$

and $\hat{\gamma} = \frac{1}{2}$. With $\zeta(\gamma) := 1 - \frac{1}{4}\gamma^{-2}$, the Hamiltonian matrices (7) and (8) become

$$H(\gamma) = \left[\begin{array}{cc|cc} -1 & -\beta & -\zeta(\gamma) & -\zeta(\gamma) \\ 0 & -1-\beta & -\zeta(\gamma) & -\zeta(\gamma) \\ \hline -\frac{\alpha^2}{\zeta(\gamma)} & 0 & 1 & 0 \\ 0 & 0 & \beta & 1+\beta \end{array}\right]$$

and

$$J(\gamma) = \left[\begin{array}{cc|cc} -1 & 0 & \alpha^2\gamma^{-2}\zeta^{-1}(\gamma) - \delta^2\zeta(\gamma) & -\frac{\beta\delta}{2}\gamma^{-2} - \delta\eta\zeta(\gamma) \\ 0 & -1 & -\frac{\beta\delta}{2}\gamma^{-2} - \delta\eta\zeta(\gamma) & (\beta-\eta)\beta\gamma^{-2} - \eta^2\zeta(\gamma) \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

The matrix $J(\gamma)$ has a double eigenvalue -1 and the corresponding positive semidefinite ARE solution is $X_J = 0$. The matrix $H(\gamma)$ has the two eigenvalues -1 and $-\sqrt{(1+\beta)^2+\alpha^2}$ in the open left half complex plane. When $\gamma > \frac{1}{2}$, setting $\nu := 1 + \sqrt{(1+\beta)^2+\alpha^2}$, the positive semidefinite ARE solution corresponding to $H(\gamma)$ is

$$X_H = \frac{\alpha^2}{\zeta(\gamma)(2\beta+\beta^2+\alpha^2)} \cdot \begin{bmatrix} \frac{\beta(2+\beta)}{2} + \frac{\alpha^2}{\nu} & \beta(2+\beta)\left(\frac{1}{\nu} - \frac{1}{2}\right) \\ \beta(2+\beta)\left(\frac{1}{\nu} - \frac{1}{2}\right) & \beta^2\left(\frac{1}{2} - \frac{(2+\beta)}{\nu(\nu+\beta)}\right) \end{bmatrix}.$$

If $\beta^2 + 2\beta + \alpha^2 = 0$, then

$$X_H = \frac{\alpha^2}{8\zeta(\gamma)} \begin{bmatrix} 4-\alpha^2 & \alpha^2 \\ \alpha^2 & \beta^2\left(1 + \frac{2}{2+\beta}\right) \end{bmatrix}.$$

Note that in this case $|\alpha| \leq 1$ and $\beta = -1 \pm \sqrt{1-\alpha^2}$. Moreover, $H(\gamma)$ has the double eigenvalues 1 and -1.

Since the semi-stabilizing ARE solutions X_H and X_J exist and $\rho(X_J X_H) = 0$ for all $\gamma > \hat{\gamma}$, we have $\gamma_{mo} = \hat{\gamma} = \frac{1}{2}$. As γ approaches γ_{mo}, the function $\zeta(\gamma)$ approaches 0. The matrices R_H and R_J become singular, the Hamiltonian matrices $H(\gamma)$ and $J(\gamma)$ become ill-defined, and the ARE solution X_H converges to infinity.

Typical numerical ARE solvers are unable to succeed on problems as extreme as those in Example 3.1 with $\gamma \approx \hat{\gamma}$. Failing to solve an ARE may cause a computation to abort before attaining a close approximation to γ_{mo} [29]. In the most extreme case, $H(\gamma)$, $J(\gamma)$ or the associated ARE solution may have entries larger than the overflow threshold and may not be representable in the working floating point number system, thus leading to the failure of any numerical method that explicitly constructs any of the matrices in Theorem 2.7!

A more subtle and more likely problem (also observed in [26]) is that explicitly forming the Hamiltonian matrices themselves may lead to large inaccuracies. If the matrices $R_H(\gamma)$ or $R_J(\gamma)$ are ill-conditioned or if cancellation errors occur in computing the blocks of $H(\gamma)$ and $J(\gamma)$, then the input data for the ARE solvers may be corrupted. Example 3.1 demonstrates how the matrices $R_H(\gamma)$ and $R_J(\gamma)$ become nearly singular and highly ill-conditioned as γ approaches $\hat{\gamma} = \gamma_{mo}$.

As suggested in [19, 44, 26], a suitable embedding of the Hamiltonian matrices into matrix pencils may avoid problems caused by explicitly forming the Hamiltonian matrices. In Section 4, we will discuss a new variant of this approach which is similar to the structured embedding technique introduced in [4]. The method discussed in [26] avoids much of the trouble with the Riccati method, but it has some drawbacks. First of all, it computes explicit solutions to the AREs. Also, since it uses the general QZ algorithm to compute deflating subspaces, it does not make use or preserve the special structure of the eigenvalue problem. This becomes critical when there are eigenvalues close to or on the imaginary axis as may happen near γ_{mo}. Unstructured numerical methods are not reliable when there are eigenvalues on or near the imaginary axis as roundoff errors may cause stable eigenvalues to become computed unstable eigenvalues, see [25]. In such a situation, it is likely that a wrong decision is taken by the γ-iteration.

To facilitate our discussion, we introduce some notation for several critical points of γ that play a role in determining γ_{mo}.

Definition 3.2 Define $\hat{\gamma}_H^R$, $\hat{\gamma}_J^R$ and $\hat{\gamma}^R = \max(\hat{\gamma}_H^R, \hat{\gamma}_J^R)$ by

$$\begin{aligned}
\hat{\gamma}_H^R &= \inf\left\{\gamma \geq \hat{\gamma} \;\middle|\; \begin{array}{l}\text{The ARE corresponding to (7) has a positive}\\ \text{semi-definite, semi-stabilizing solution.}\end{array}\right\},\\
\hat{\gamma}_J^R &= \inf\left\{\gamma \geq \hat{\gamma} \;\middle|\; \begin{array}{l}\text{The ARE corresponding to (8) has a positive}\\ \text{semi-definite, semi-stabilizing solution.}\end{array}\right\}.
\end{aligned}$$

Definition 3.3 Define $\hat{\gamma}_H^L$, $\hat{\gamma}_J^L$ and $\hat{\gamma}^L = \max(\hat{\gamma}_H^L, \hat{\gamma}_J^L)$ by

$$\begin{aligned}
\hat{\gamma}_H^L &= \inf\left\{\gamma \geq \hat{\gamma} \;\middle|\; \begin{array}{l}\text{The Hamiltonian matrix } H(\gamma) \text{ in (7) has a}\\ \text{semi-stable Lagrangian invariant subspace.}\end{array}\right\},\\
\hat{\gamma}_J^L &= \inf\left\{\gamma \geq \hat{\gamma} \;\middle|\; \begin{array}{l}\text{The Hamiltonian matrix } J(\gamma) \text{ in (8) has a}\\ \text{semi-stable Lagrangian invariant subspace.}\end{array}\right\}.
\end{aligned}$$

Definition 3.4 Define $\hat{\gamma}_H^I$, $\hat{\gamma}_J^I$ and $\hat{\gamma}^I = \max(\hat{\gamma}_H^I, \hat{\gamma}_J^I)$ by

$$\begin{aligned}
\hat{\gamma}_H^I &= \sup\left\{\gamma > \hat{\gamma} \;\middle|\; \begin{array}{l}\text{The Hamiltonian matrix } H(\gamma) \text{ in (7) has an}\\ \text{eigenvalue on the imaginary axis.}\end{array}\right\},\\
\hat{\gamma}_J^I &= \sup\left\{\gamma > \hat{\gamma} \;\middle|\; \begin{array}{l}\text{The Hamiltonian matrix } J(\gamma) \text{ in (8) has an}\\ \text{eigenvalue on the imaginary axis.}\end{array}\right\}.
\end{aligned}$$

If both $H(\gamma), J(\gamma)$ have no eigenvalues on the imaginary axis for all $\gamma > \hat{\gamma}$, then $\hat{\gamma}^I$ does not exist. Note that under Assumptions A1-A4, $\hat{\gamma}$, $\hat{\gamma}^L$ and $\hat{\gamma}^R$ satisfy $0 \leq \hat{\gamma} \leq \hat{\gamma}^L \leq \hat{\gamma}^R$. If $\hat{\gamma}^I$ exists, then $\hat{\gamma}^I = \hat{\gamma}^L > \hat{\gamma}$.

If $\gamma = \hat{\gamma}^I$, then one or both of the Hamiltonian matrices $H(\gamma)$ or $J(\gamma)$ have eigenvalues on the imaginary axis. Even with otherwise robust numerical methods like the QR algorithm, rounding errors made while calculating eigenvalues and invariant subspaces may introduce non-Hamiltonian perturbations of the Hamiltonian matrix. Unstructured, non-Hamiltonian rounding errors may destroy the uniqueness of the semi-stable Lagrangian invariant subspace [42, 43] causing any Riccati solver to fail. Even the number of eigenvalues in the closed left-half plane may drop below its theoretical value of n.

Many Riccati equation solvers begin their work by extracting the stable invariant subspace of a Hamiltonian matrix [3, 34, 37, 46]. A naive algorithm may in this case select an incorrect invariant subspace and either conclude that there is no solution to the Riccati equation or simply return a far-from-symmetric and/or non-stabilizing solution. A Hamiltonian perturbation of H will not create confusion, because the $\pm$ pairing of eigenvalues is preserved. For Hamiltonian matrices, numerically stable algorithms that fully exploit the Hamiltonian structure are derived in [9, 18].

Some of the problems discussed above are illustrated in the following example.

Example 3.5 Consider Example 3.1 with $\alpha = \beta = \delta = \eta = \epsilon_2 = 1$ and $\epsilon_1 = 0$. In this case, the Riccati solution associated with (8) has semi-stabilizing solution $X_J = 0$, independent of γ. We constructed $H(\gamma)$ in (7) for 91 values of γ equally spaced in the interval $[0.1, 1]$ and used the MATLAB builtin function `eig` (based on the QR algorithm, see, e.g., [27]) to calculate the eigenvalues of each $H(\gamma)$. In no case did any computed eigenvalue have zero real part. If a Hamiltonian matrix has no eigenvalue with zero real part, then there is a unique stabilizing solution of the corresponding algebraic Riccati equation. A naive program to calculate γ_{mo} might use this result to conclude that the algebraic Riccati equation corresponding to each

$H(\gamma)$ has a stabilizing solution for $\gamma \in [.5, 1]$. Such a program might even construct "solutions" X_H, calculate $\rho(X_H X_J) = 0$ and ultimately conclude that $\gamma_{mo} = \hat{\gamma} = 1/2$.

In fact, $\gamma_{mo} = \hat{\gamma}^I \approx 0.806$. In this example, the algebraic Riccati equation corresponding to (7) has a stabilizing positive semidefinite solution if and only if $\gamma > \hat{\gamma}^I$. As γ approaches $\hat{\gamma}^I$, a $\pm\lambda$ pair of real eigenvalues of the Hamiltonian matrix $H(\gamma)$ in (7) coalesces into a double eigenvalue at 0 corresponding to a 2-by-2 Jordan block. As γ decreases further, this double eigenvalue splits into two complex conjugate eigenvalues with zero real part.

Rounding errors constructing $H(\gamma)$ and computing its eigenvalues perturb eigenvalues off the imaginary axis. If these rounding errors are of magnitude ε then the eigenvalues of the 2-by-2 Jordan block are perturbed by $O(\sqrt{\varepsilon})$. Similar eigenvalue perturbations result from perturbations of γ near $\hat{\gamma}^I$. Thus, eigenvalues may be relatively distant from the imaginary axis even when $\hat{\gamma} \approx \hat{\gamma}^I$. Consequently, it is problematic to use the computed eigenvalues to determine whether $H(\gamma)$ has eigenvalues with zero real part and whether the corresponding algebraic Riccati equation has a stabilizing solution.

For further discussion on the benefits of structure-exploitation and mishaps caused by unstructured methods see [7, 10, 11, 24, 32].

Finally, we introduce another quantity characterizing critical cases related to the spectral radius condition in Theorem 2.7.

Definition 3.6 Let $X_H = X_H(\gamma)$, $X_J = X_J(\gamma)$ be the positive semi-definite stabilizing solutions of the Riccati equations associated with $H(\lambda)$ and $J(\lambda)$ in Theorem 2.7, respectively. Define $\hat{\gamma}^\rho$ as the largest number $\gamma \geq \hat{\gamma}$ satisfying $\gamma^2 = \rho(X_H X_J)$.

If no such number γ exists, then $\hat{\gamma}^\rho$ does not exist. If $\hat{\gamma}^\rho$ exists, then $\hat{\gamma}^\rho \geq \hat{\gamma}^R$.

We conclude this section by noting that the solution to the modified optimal H_∞ control problem is determined by the quantities introduced in this section:

$$\gamma_{mo} = \max(\hat{\gamma}, \hat{\gamma}^L, \hat{\gamma}^R, \hat{\gamma}^\rho).$$

Table 1 summarizes the definitions of the different $\hat{\gamma}$'s.

4 Reformulations

In this section we review the properties of Lagrangian invariant subspaces and Riccati solutions associated with $H(\gamma)$ and $J(\gamma)$ along with the relationship between γ_{mo} and the various $\hat{\gamma}$'s. This section also reformulates Theorem 2.7 in order to overcome numerical difficulties.

4.1 Avoiding explicit solution of Riccati equations

The solution of the algebraic Riccati equations is only an intermediate step toward solving the H_∞ control problem. Avoiding explicit solution of algebraic Riccati equations is the only way to avoid numerical instabilities like those in Example 3.1. A similar situation occurs in H_2 control problems. There, the solution of algebraic Riccati equations is an intermediate step toward the closed-loop matrix and optimal feedback. Explicit Riccati solutions may be avoided by computing deflating subspaces of matrix pencils [4, 5, 51].

The following reformulation of conditions 2 and 3 in Theorem 2.7 suggested in [54, Theorem 16.4, p. 419] employs this idea.

$\hat{\gamma}$	$\sup\{\gamma \geq 0 \mid$ either R_H or R_J is singular. $\}$
$\hat{\gamma}^L$	$\inf\{\gamma \geq \hat{\gamma} \mid$ Both $H(\gamma)$ and $J(\gamma)$ have a semi-stable Lagrangian invariant subspace.$\}$
$\hat{\gamma}^R$	$\inf\{\gamma \geq \hat{\gamma} \mid$ Both $X_H(\gamma)$ and $X_J(\gamma)$ exist.$\}$
$\hat{\gamma}^I$	$\sup\left\{\gamma > \hat{\gamma} \,\middle\vert\, \begin{array}{l}\text{Either } H(\gamma) \text{ or } J(\gamma) \text{ has an eigenvalue with zero real} \\ \text{part. Note that } \hat{\gamma}^I \text{ may or may not exist.}\end{array}\right\}$
$\hat{\gamma}^\rho$	Largest number $\gamma \geq \hat{\gamma}$ satisfying $\gamma^2 = \rho(X_H X_J)$. Note that $\hat{\gamma}^\rho$ may or may not exist.

Table 1: Summary of definitions of the $\hat{\gamma}$'s. Here $X_H = X_H(\gamma)$ and $X_J = X_J(\gamma)$ are the positive semi-definite stabilizing solutions of the Riccati equations corresponding to the Hamiltonian matrices $H(\gamma)$ and $J(\gamma)$ in Theorem 2.7. The $\hat{\gamma}$'s satisfy $0 \leq \hat{\gamma} \leq \hat{\gamma}^L \leq \hat{\gamma}^R$. If $\hat{\gamma}^I$ exists, $\hat{\gamma}^I = \hat{\gamma}^L > \hat{\gamma}$. If $\hat{\gamma}^\rho$ exists, then $\hat{\gamma}^\rho \geq \hat{\gamma}^R$.

2'. There exist matrices $X_{H,1}$, $X_{H,2} \in \mathbb{R}^{n,n}$ with $X_{H,1}$ nonsingular such that the columns of $\begin{bmatrix} X_{H,1} \\ X_{H,2} \end{bmatrix}$, span a semi-stable Lagrangian invariant subspace of $H(\gamma)$, i.e., there exists a semi-stable matrix T_H for which

$$H(\gamma) \begin{bmatrix} X_{H,1} \\ X_{H,2} \end{bmatrix} = \begin{bmatrix} X_{H,1} \\ X_{H,2} \end{bmatrix} T_H. \tag{9}$$

3'. There exist matrices $X_{J,1}, X_{J,2} \in \mathbb{R}^{n,n}$ with $X_{J,1}$ nonsingular such that the columns of $\begin{bmatrix} X_{J,1} \\ X_{J,2} \end{bmatrix}$, span a semi-stable Lagrangian invariant subspace of $J(\gamma)$, i.e., there exists a semi-stable matrix T_J for which

$$J(\gamma) \begin{bmatrix} X_{J,1} \\ X_{J,2} \end{bmatrix} = \begin{bmatrix} X_{J,1} \\ X_{J,2} \end{bmatrix} T_J. \tag{10}$$

(Below, we will further reformulate the invariant subspace approach and remove the non-singularity requirement for $X_{H,1}$ and $X_{J,1}$.)

The reformulation by the alternative conditions $2'$ and $3'$ is helpful, because when $\gamma = \hat{\gamma}_H^I$ (or $\gamma = \hat{\gamma}_J^I$), then $H(\gamma)$ (or $J(\gamma)$) may have a unique semi-stable Lagrangian subspace but no positive semi-stabilizing solution to the associated Riccati equation. Furthermore, there exist Hamiltonian matrices for which the computation of the unique semi-stable Lagrangian invariant subspace is well-conditioned, but the solution of the Riccati equation is ill-conditioned, see e.g. [7, 6].

4.2 Avoiding the spectral radius condition

In order to avoid explicit Riccati solutions entirely, we must also reformulate the 4th condition of Theorem 2.7, $\rho(X_H X_J) < \gamma^2$, in terms of the semi-stable, Lagrangian invariant subspaces (9) and (10). See also [54, Section 16.11] and [44]. Here, we propose a new formulation that requires solely numerically backward stable matrix factorizations.

Let $\begin{bmatrix} X_{H,1}(\gamma) \\ X_{H,2}(\gamma) \end{bmatrix}$ and $\begin{bmatrix} X_{J,1}(\gamma) \\ X_{J,2}(\gamma) \end{bmatrix}$ be as in (9) and (10), respectively, and set

$$\mathcal{Y}(\gamma) := \begin{bmatrix} \gamma X_{H,2}^T X_{H,1} & X_{H,2}^T X_{J,2} \\ X_{J,2}^T X_{H,2} & \gamma X_{J,2}^T X_{J,1} \end{bmatrix}. \tag{11}$$

Note that all the blocks of $\mathcal{Y}$ are functions of γ, even if γ does not appear explicitly in the off-diagonal blocks. If one of the semi-stable, Lagrangian invariant subspaces does not exist, then $\mathcal{Y}(\gamma)$ is undefined. We will show that $\mathcal{Y}(\gamma)$ is positive semidefinite with a particular rank if and only if the Riccati solutions X_H and X_J in Theorem 2.7 exist and $\gamma^2 > \rho(X_H X_J)$.

Theorem 4.1 *[22, 31, 53] Under assumptions A1-A4, γ_{mo} exists. The solutions $X_H = X_H(\gamma)$ and $X_J = X_J(\gamma)$ of the algebraic Riccati equations associated with $H(\gamma)$ and $J(\gamma)$ in (7) and (8) as well as the spectral radius $\rho(X_H X_J) = \rho(X_H(\gamma)X_J(\gamma))$ are monotonically decreasing functions of γ on the infinite interval $\mathcal{I} = (\gamma_{mo}, \infty)$, i.e., if $\gamma_{mo} < \gamma_1 \leq \gamma_2$, then $X_H(\gamma_2) \leq X_H(\gamma_1)$, $X_J(\gamma_2) \leq X_J(\gamma_1)$ and $\rho(X_H(\gamma_2)X_J(\gamma_2)) \leq \rho(X_H(\gamma_1)X_J(\gamma_1))$. In addition, the ranks of $X_H = X_H(\gamma)$ and $X_J = X_J(\gamma)$ are constant on $\mathcal{I}$.*

Proof. See [22, 53]. For particularly complete proofs, see [31, Theorems 2.4, 4.1, 5.1]. □

The following well-known theorem on the *CS decomposition* of orthonormal bases of Lagrangian subspaces helps display the internal structure of $\mathcal{Y}(\gamma)$.

Lemma 4.2 [39] *If X_1, $X_2 \in \mathbb{R}^{n,n}$ and the columns of $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ form an orthonormal basis of a Lagrangian subspace, then there exist orthogonal matrices $U \in \mathbb{R}^{n,n}$ and $V \in \mathbb{R}^{n,n}$ such that $U^T X_1 V = C$ and $U^T X_2 V = S$ are both diagonal and $C^2 + S^2 = I$.*

Apply Lemma 4.2 to $\begin{bmatrix} X_{H,1}(\gamma) \\ X_{H,2}(\gamma) \end{bmatrix}$, $\begin{bmatrix} X_{J,1}(\gamma) \\ X_{J,2}(\gamma) \end{bmatrix}$ separating diagonal elements of C and S that equal zero or one, we get

$$U_H^T X_{H,1} V_H = C_H =: \begin{array}{c} \\ r_H \\ k_H \\ n - t_H \end{array} \begin{array}{c} \begin{array}{ccc} r_H & k_H & n - t_H \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Sigma_H & 0 \\ 0 & 0 & I \end{bmatrix} \end{array}, \tag{12}$$

$$U_H^T X_{H,2} V_H = S_H =: \begin{array}{c} \\ r_H \\ k_H \\ n - t_H \end{array} \begin{array}{c} \begin{array}{ccc} r_H & k_H & n - t_H \end{array} \\ \begin{bmatrix} I & 0 & 0 \\ 0 & \Delta_H & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}, \tag{13}$$

$$U_J^T X_{J,1} V_J = C_J =: \begin{array}{c} \\ r_J \\ k_J \\ n - t_J \end{array} \begin{array}{c} \begin{array}{ccc} r_J & k_J & n - t_J \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Sigma_J & 0 \\ 0 & 0 & I \end{bmatrix} \end{array}, \tag{14}$$

$$U_J^T X_{J,2} V_J = S_J \quad =: \quad \begin{array}{c} \\ r_J \\ k_J \\ n-t_J \end{array} \begin{array}{c} \begin{array}{ccc} r_J & k_J & n-t_J \end{array} \\ \begin{bmatrix} I & 0 & 0 \\ 0 & \Delta_J & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}, \tag{15}$$

where $k_H + r_H = t_H$, $k_J + r_J = t_J$, Σ_H, Δ_H, Σ_J and Δ_J are diagonal, nonsingular and satisfy $\Sigma_H^2 + \Delta_H^2 = I$ and $\Sigma_J^2 + \Delta_J^2 = I$. If $r_H = r_J = 0$, then $k_H = t_H$, $k_J = t_J$. In this case both X_H, X_J exist and

$$U_H^T X_H U_H \quad = \quad \begin{array}{c} \\ k_H \\ n-k_H \end{array} \begin{array}{c} \begin{array}{cc} k_H & n-k_H \end{array} \\ \begin{bmatrix} \Delta_H \Sigma_H^{-1} & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \tag{16}$$

$$U_J^T X_J U_J \quad = \quad \begin{array}{c} \\ k_J \\ n-k_J \end{array} \begin{array}{c} \begin{array}{cc} k_J & n-k_J \end{array} \\ \begin{bmatrix} \Delta_J \Sigma_J^{-1} & 0 \\ 0 & 0 \end{bmatrix} \end{array}. \tag{17}$$

Also, Theorem 4.1 shows that $k_H = \operatorname{rank} X_H$ and $k_J = \operatorname{rank} X_J$ are constant for $\gamma > \gamma_{mo}$. Define $\tilde{\mathcal{Y}}(\gamma)$ by

$$\tilde{\mathcal{Y}}(\gamma) \quad = \quad \begin{bmatrix} V_H^T & 0 \\ 0 & V_J^T \end{bmatrix} \mathcal{Y}(\gamma) \begin{bmatrix} V_H & 0 \\ 0 & V_J \end{bmatrix} \tag{18}$$

$$= \begin{array}{c} \\ r_H \\ k_H \\ n-t_H \\ r_J \\ k_J \\ n-t_J \end{array} \begin{array}{c} \begin{array}{cccccc} r_H & k_H & n-t_H & r_J & k_J & n-t_J \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & Q_{11} & Q_{12}\Delta_J & 0 \\ 0 & \gamma\Delta_H\Sigma_H & 0 & \Delta_H Q_{21} & \Delta_H Q_{22}\Delta_J & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ Q_{11}^T & Q_{21}^T\Delta_H & 0 & 0 & 0 & 0 \\ \Delta_J Q_{12}^T & \Delta_J Q_{22}^T \Delta_H & 0 & 0 & \gamma\Delta_J\Sigma_J & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

where the blocks Q_{11}, Q_{12}, Q_{21}, and Q_{22} are sub-blocks of the orthogonal matrix

$$U_H^T U_J = \begin{array}{c} \\ r_H \\ k_H \\ n-t_H \end{array} \begin{array}{c} \begin{array}{ccc} r_J & k_J & n-t_J \end{array} \\ \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \end{array}. \tag{19}$$

The following lemma shows the relationship between $\mathcal{Y}(\gamma)$, X_H, X_J, and $\gamma^2 - \rho(X_H X_J)$.

Lemma 4.3 *Let $\hat{k}_H = \operatorname{rank} X_H(\gamma_0)$ and $\hat{k}_J = \operatorname{rank} X_J(\gamma_0)$ for some $\gamma_0 > \gamma_{mo}$.*

i) $\mathcal{Y}(\gamma) \geq 0$ if and only if each of the blocks Q_{11}, Q_{12}, Q_{21} in (19) are either zero or void and

$$\begin{bmatrix} \gamma\Delta_H\Sigma_H & \Delta_H Q_{22}\Delta_J \\ \Delta_J Q_{22}^T \Delta_H & \gamma\Delta_J\Sigma_J \end{bmatrix} \geq 0.$$

ii) $\mathcal{Y}(\gamma) \geq 0$ and $\operatorname{rank}\mathcal{Y}(\gamma) = \hat{k}_H + \hat{k}_J$ if and only if the (semi-)stabilizing, positive semidefinite Riccati solutions X_H and X_J in Theorem 2.7 exist and $\gamma^2 > \rho(X_H X_J)$.

Proof. The matrix $\mathcal{Y}(\gamma)$ is a congruence transformation of $\tilde{\mathcal{Y}}(\gamma)$ in (18). Hence $\mathcal{Y}(\gamma)$ is positive (semi)definite if and only if $\tilde{\mathcal{Y}}$ is positive (semi)definite. Statement *i)* now follows immediately from (18).

ii) Note that $n - t_H$ and $n - t_j$ are independent of γ, [31]. By Theorem 4.1, it is clear that $\hat{k}_H = n - t_H$ and $\hat{k}_J = n - t_J$. If $\mathcal{Y}(\gamma) \geq 0$ and $\operatorname{rank} \mathcal{Y}(\gamma) = \hat{k}_H + \hat{k}_J$, then it follows from (18) that $r_H = r_J = 0$. If $X_H = X_H(\gamma)$ and $X_J = X_J(\gamma)$ exist, then it follows from (12)–(15) that $r_H = r_J = 0$. So, in either the forward hypothesis of Statement *ii)* or the converse hypothesis, it holds that $r_H = r_J = 0$ and that Q_{11}, Q_{12} and Q_{21} are void.

Using (16) and (17), the product $X_H X_J$ can be written as

$$\begin{aligned} U_J^T X_H X_J U_J &= U_J^T U_H \begin{bmatrix} \Sigma_H^{-1}\Delta_H & 0 \\ 0 & 0 \end{bmatrix} U_H^T U_J \begin{bmatrix} \Sigma_J^{-1}\Delta_J & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} Q_{22}^T \Sigma_H^{-1}\Delta_H Q_{22} \Sigma_J^{-1}\Delta_J & 0 \\ Q_{23}^T \Sigma_H^{-1}\Delta_H Q_{22} \Sigma_J^{-1}\Delta_J & 0 \end{bmatrix}. \end{aligned}$$

Hence, $\rho(X_H X_J) = \rho(Q_{22}^T \Sigma_H^{-1}\Delta_H Q_{22} \Sigma_J^{-1}\Delta_J)$, and

$$\begin{aligned} &\gamma^2 - \rho(X_H X_J) > 0 \\ &\iff \gamma^2 - \rho(Q_{22}^T \Sigma_H^{-1}\Delta_H Q_{22} \Sigma_J^{-1}\Delta_J) > 0 \\ &\iff \gamma^2 - \rho((\Sigma_J^{-1}\Delta_J)^{\frac{1}{2}} Q_{22}^T \Sigma_H^{-1}\Delta_H Q_{22} (\Sigma_J^{-1}\Delta_J)^{\frac{1}{2}}) > 0 \\ &\iff \gamma^2 I - (\Sigma_J^{-1}\Delta_J)^{\frac{1}{2}} Q_{22}^T \Sigma_H^{-1}\Delta_H Q_{22} (\Sigma_J^{-1}\Delta_J)^{\frac{1}{2}} > 0 \\ &\iff \gamma^2 \Sigma_J^{-1}\Delta_J - \Sigma_J^{-1}\Delta_J Q_{22}^T \Sigma_H^{-1}\Delta_H Q_{22} \Sigma_J^{-1}\Delta_J > 0. \end{aligned} \tag{20}$$

The matrix $\tilde{\mathcal{Y}}(\gamma)$ factors as

$$\tilde{\mathcal{Y}}(\gamma) = T \left[\begin{array}{cc|cc} \gamma\Sigma_H^{-1}\Delta_H & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Y_{33} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] T^T, \tag{21}$$

where

$$Y_{33} = \gamma\Sigma_J^{-1}\Delta_J - \gamma^{-1}\Sigma_J^{-1}\Delta_J Q_{22}^T \Delta_H \Sigma_H^{-1} Q_{22} \Delta_J \Sigma_J^{-1},$$

and

$$T = \left[\begin{array}{cc|cc} \Sigma_H & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline \gamma^{-1}\Delta_J Q_{22}^T & 0 & \Sigma_J & 0 \\ 0 & 0 & 0 & I \end{array}\right].$$

Hence,

$$\begin{aligned} &\mathcal{Y}(\gamma) \geq 0, \quad \text{and} \quad \operatorname{rank} \mathcal{Y} = \hat{k}_H + \hat{k}_J \\ &\iff \tilde{\mathcal{Y}}(\gamma) \geq 0, \quad \text{and} \quad \operatorname{rank} \tilde{\mathcal{Y}} = \hat{k}_H + \hat{k}_J \\ &\iff \Sigma_H^{-1}\Delta_H > 0, \quad \text{and} \quad \Sigma_J^{-1}\Delta_J > 0, \quad \text{and} \quad Y_{33} > 0 \\ &\iff X_H \geq 0 \quad \text{and} \quad X_J \geq 0 \quad \text{and} \\ &\qquad \gamma^2 I - (\Sigma_J^{-1}\Delta_J)^{\frac{1}{2}} Q_{22}^T \Delta_H \Sigma_H^{-1} Q_{22} (\Delta_J \Sigma_J^{-1})^{\frac{1}{2}} > 0 \\ &\iff X_H \geq 0 \quad \text{and} \quad X_J \geq 0 \quad \text{and} \quad \gamma^2 - \rho(X_H X_J) > 0. \end{aligned}$$

□

In summary, the problem of finding γ_{mo} reduces to the problem of finding the largest value of $\gamma \geq \hat{\gamma}$ at which $\mathcal{Y}(\gamma) \geq 0$ does not hold, or $\mathcal{Y}(\gamma)$ changes rank or fails to exist. The following theorem summarizes these observations.

Theorem 4.4 *For all $\gamma > \gamma_{mo}$, $\mathcal{Y}(\gamma) \geq 0$ and $\operatorname{rank} \mathcal{Y}(\gamma) = \hat{k}_H + \hat{k}_J$ is constant. For all $\hat{\gamma} < \gamma < \gamma_{mo}$, either $\mathcal{Y}(\gamma)$ is not defined, or $\operatorname{rank} \mathcal{Y}(\gamma) < \hat{k}_H + \hat{k}_J$, or $\mathcal{Y}(\gamma)$ is not positive semidefinite.*

Example 4.5 Returning to Example 3.1, observe that checking the semi-definiteness of X_H and X_J and the spectral radius $\rho(X_J X_H)$ may not be a viable procedure as γ approaches γ_{mo}, because $X_H = X_H(\gamma)$ diverges to infinity. In contrast, $\mathcal{Y}(\gamma)$ and $\tilde{\mathcal{Y}}(\gamma)$ remain bounded as γ approaches γ_{mo}. Using the CS decomposition to check the rank of $\tilde{\mathcal{Y}}(\gamma)$ is reliable as long as orthogonal bases of the semi-stable Lagrangian invariant subspaces are computed accurately.

Remark 4.6 Theorem 4.1 states that $X_H = X_H(\gamma)$, $X_J = X_J(\gamma)$ and $\rho(X_H X_J)$ are monotone in γ. However, neither $\mathcal{Y}(\gamma)$ nor $\tilde{\mathcal{Y}}(\gamma)$ are monotone in γ, see Figure 1.

Remark 4.7 Let $f(\gamma)$ be the $(\hat{k}_H + \hat{k}_J)$-th largest eigenvalue of $\mathcal{Y}(\gamma)$. Theorem 4.4 shows that γ_{mo} is often the largest root of $f(\gamma)$. In principle, rapidly convergent one dimensional root finding methods can be applied. However, it is our observation that the paths of the eigenvalues of $\mathcal{Y}(\gamma)$ often intersect near γ_{mo}, thus creating a discontinuity in the first derivative of $f(\gamma)$, see Figure 1. Consequently, rapidly converging methods like the secant method accelerate convergence only after a more slowly converging method like bisection has already attained a good approximation to γ_{mo}.

4.3 Avoiding R_H^{-1} and R_J^{-1}

The formulas (7) and (8) of the Hamiltonian matrices $H(\gamma)$ and $J(\gamma)$ involve inverses of matrices that may be ill-conditioned along with many matrix products and matrix sums that may involve subtractive cancellation of significant digits. The Hamiltonian matrices constructed in the presence of finite precision arithmetic may become so corrupted by rounding errors that accurate calculation of the semi-stable invariant subspaces is impossible, see Example 3.1.

In order to avoid these difficulties we employ a structured version of the embedding introduced in [19, 26]. Here, we embed the Hamiltonian eigenvalue problems into so called *even generalized eigenvalue problems*, see [15, 36], which generalize Hamiltonian matrices and skew-Hamiltonian/Hamiltonian pencils, see [4].

Definition 4.8 *A matrix pencil $\lambda N - M$ with $N, M \in \mathbb{R}^{2n+r,2n+r}$ is called* even *if $N = -N^T$ and $M = M^T$.*

Real even matrix pencils have Hamiltonian eigensymmetry, i.e. the finite eigenvalues are symmetric about both the real axis and the imaginary axis, see [36]. The analysis of even pencils and appropriate Kronecker and staircase forms have been presented in [15, 49].

Set $r = m_1 + m_2 + p_1$, $\tilde{r} = m_1 + p_1 + p_2$ and form the pencils

$$\lambda N - M_H(\gamma) := \lambda \left[\begin{array}{cc|ccc} 0 & I_n & 0 & 0 & 0 \\ -I_n & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{cc|ccc} 0 & -A^T & 0 & 0 & -C_1^T \\ -A & 0 & B_1 & B_2 & 0 \\ \hline 0 & B_1^T & \gamma^2 I_{m_1} & 0 & D_{11}^T \\ 0 & B_2^T & 0 & 0 & D_{12}^T \\ -C_1 & 0 & D_{11} & D_{12} & I_{p_1} \end{array}\right] \tag{22}$$

and

$$\lambda N - M_J(\gamma) := \lambda \left[\begin{array}{cc|ccc} 0 & I_n & 0 & 0 & 0 \\ -I_n & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{cc|ccc} 0 & -A & 0 & 0 & -B_1 \\ -A^T & 0 & C_1^T & C_2^T & 0 \\ \hline 0 & C_1 & \gamma^2 I_{p_1} & 0 & D_{11} \\ 0 & C_2 & 0 & 0 & D_{21} \\ -B_1^T & 0 & D_{11}^T & D_{21}^T & I_{m_1} \end{array}\right] \tag{23}$$

Let us consider the pencils $\lambda N - M_H$ and $\lambda N - M_J$ in more detail.

Proposition 4.9 *i) The pencil $\lambda N - M_H(\gamma)$ is regular and of index at most one if and only if $R_H(\gamma)$ is invertible. In this case, $\lambda N - M_H(\gamma)$ has exactly $2n$ finite eigenvalues.*

ii) The pencil $\lambda N - M_J(\gamma)$ is regular and of index at most one if and only if $R_J(\gamma)$ is invertible. In this case, $\lambda N - M_J(\gamma)$ has exactly $2n$ finite eigenvalues.

Proof. See, e.g. [15, 37]. □

This leads us to a characterization for the existence and uniqueness of deflating subspaces.

Theorem 4.10 *Suppose that the assumptions A1–A4 are satisfied.*

i) If $\hat{\gamma}_H^I$ exists, then for all $\gamma > \hat{\gamma}_H^I$ the even pencil $\lambda N - M_H(\gamma)$ has a unique n-dimensional stable deflating subspace. At $\gamma = \hat{\gamma}_H^I$, $\lambda N - M_H(\hat{\gamma}_H^I)$ has a unique n-dimensional semi-stable deflating subspace.

If $\hat{\gamma}_H^I$ does not exist, then for all $\gamma > \hat{\gamma}_H$, $\lambda N - M_H(\gamma)$ has a unique n-dimensional stable deflating subspace.

ii) If $\hat{\gamma}_J^I$ exists, then for all $\gamma > \hat{\gamma}_J^I$ the even pencil $\lambda N - M_J(\gamma)$ has a unique n-dimensional stable deflating subspace. At $\gamma = \hat{\gamma}_J^I$, $\lambda N - M_J(\hat{\gamma}_J^I)$ has a unique n-dimensional semi-stable deflating subspace.

If $\hat{\gamma}_J^I$ does not exist, then for all $\gamma > \hat{\gamma}_J$, $\lambda N - M_J(\gamma)$ has a unique n-dimensional stable deflating subspace.

Furthermore, if

$$Q_H = \left[\begin{array}{c} Q_{H,1} \\ Q_{H,2} \\ Q_{H,3} \\ Q_{H,4} \\ Q_{H,5} \end{array}\right] \in \mathbb{R}^{2n+r,n}, \qquad Q_J = \left[\begin{array}{c} Q_{J,1} \\ Q_{J,2} \\ Q_{J,3} \\ Q_{J,4} \\ Q_{J,5} \end{array}\right] \in \mathbb{R}^{2n+\tilde{r},n} \tag{24}$$

are matrices partitioned conformally with (22) and (23) and whose columns span the unique (semi-)stable deflating subspaces of $\lambda N - M_H(\gamma)$ and $\lambda N - M_J(\gamma)$ then the columns of

$$\left[\begin{array}{c} Q_{H,1} \\ Q_{H,2} \end{array}\right], \qquad \left[\begin{array}{c} Q_{J,1} \\ Q_{J,2} \end{array}\right] \tag{25}$$

span the (semi-)stable Lagrangian invariant subspaces of $H(\gamma)$ and $J(\gamma)$, respectively.

Proof. We only prove i), the proof of ii) is analogous. Since $\hat{\gamma}_H^I > \hat{\gamma}$, (when $\hat{\gamma}_H^I$ exists), the matrix $R_H(\gamma)$ and thus the pencil $\lambda N - M_H(\gamma)$ is regular and has index at most one for all $\gamma > \hat{\gamma}_H^I$. By Proposition 4.9 the pencil $\lambda N - M_H(\gamma)$ have exactly $2n$ finite eigenvalues. If $\hat{\gamma}_H^I$ does not exist, the same is true for all $\gamma > \hat{\gamma}$. Because $\lambda N - M_H(\gamma)$ is an even pencil, these finite eigenvalues have the Hamiltonian eigensymmetry. Hence there exists an n-dimensional deflating subspace associated with all eigenvalues in the open left half plane.

If the columns of Q_H span such a subspace, i.e. $M_H Q_H = N Q_H T_H$ for some matrix T_H with eigenvalues in the open left half plane, then

$$\left[\begin{array}{ccc|ccc} 0 & -A^T & 0 & 0 & -C_1^T \\ -A & 0 & B_1 & B_2 & 0 \\ \hline 0 & B_1^T & \gamma^2 I_{m_1} & 0 & D_{11}^T \\ 0 & B_2^T & 0 & 0 & D_{12}^T \\ -C_1 & 0 & D_{11} & D_{12} & I_{p_1} \end{array}\right] \left[\begin{array}{c} Q_{H,1} \\ Q_{H,2} \\ Q_{H,3} \\ Q_{H,4} \\ Q_{H,5} \end{array}\right] = \left[\begin{array}{c} Q_{H,2} \\ -Q_{H,1} \\ 0 \\ 0 \\ 0 \end{array}\right] T_H$$

Since

$$\left[\begin{array}{ccc} I_{m_1} & 0 & -D_{11}^T \\ 0 & I & -D_{12}^T \\ 0 & 0 & I_{p_1} \end{array}\right] \left[\begin{array}{ccc} \gamma^2 I_{m_1} & 0 & D_{11}^T \\ 0 & 0 & D_{12}^T \\ D_{11} & D_{12} & I_{p_1} \end{array}\right] = \left[\begin{array}{cc|c} -R_H & & 0 \\ \hline D_{11} & D_{12} & I_{p_1} \end{array}\right]$$

and R_H is invertible, we can first use the last diagonal block I_{p_1} to eliminate in the last block column and then the resulting new diagonal block R_H to obtain that

$$\left(\left[\begin{array}{cc} -C_1^T C_1 & -A^T \\ -A & 0 \end{array}\right] + \left[\begin{array}{cc} C_1^T D_{11} & C_1^T D_{12} \\ B_1 & B_2 \end{array}\right] R_H^{-1} \left[\begin{array}{cc} D_{11}^T C_1 & B_1^T \\ D_{12}^T C_1 & B_2^T \end{array}\right]\right) \left[\begin{array}{c} Q_{H,1} \\ Q_{H,2} \end{array}\right] = \left[\begin{array}{c} Q_{H,2} \\ -Q_{H,1} \end{array}\right] T_H. \tag{26}$$

A simple calculation shows that this is equivalent to (9). The same is true for the pencil $\lambda N - M_J$.

At $\hat{\gamma}_H^I$, as shown in [25], the pencil has a unique semi-stable deflating subspace. □

It follows from this theorem that in the computation of γ_{mo} it suffices to compute deflating subspaces of the even pencils in (22) and (23) associated with the closed left half plane eigenvalues. It is important that the deflating subspaces be computed with a structure preserving numerical method. It has been shown in [42, 43] that the uniqueness of a Lagrangian invariant subspace is not invariant under non-structured perturbations, see also [25]. Also, rounding errors in a non-structure preserving method may destroy the eigenvalue symmetry. In particular, if eigenvalues lie near or on the imaginary axis, rounding errors in a non-structure preserving method like the QZ algorithm may cause the numerical method to find fewer than n eigenvalues in the closed left half plane. This in turn makes it difficult or impossible to determine the desired Lagrangian invariant subspace, see, e.g., [10, 11, 24, 32] or [7, Example 4.5]. In contrast, structure-preserving methods typically compute a nearby Lagrangian subspace even when eigenvalues are near or on the imaginary axis.

Remark 4.11 The columns of $\left[\begin{smallmatrix} Q_{H,1} \\ Q_{H,2} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} Q_{J,1} \\ Q_{J,2} \end{smallmatrix}\right]$ in (25) may not be orthonormal even when the matrices Q_H and Q_J in (24) do have orthonormal columns. A numerically stable, structure preserving numerical method for extracting an orthonormal basis of a Lagrangian subspace

is the symplectic QR decomposition, see [14]. The symplectic QR decomposition determines orthogonal symplectic matrices

$$S_H = \begin{bmatrix} S_{H,1} & -S_{H,2} \\ S_{H,2} & S_{H,1} \end{bmatrix},\ S_J = \begin{bmatrix} S_{J,1} & -S_{J,2} \\ S_{J,2} & S_{J,1} \end{bmatrix},$$

such that

$$S_H \begin{bmatrix} Q_{H,1} \\ Q_{H,2} \end{bmatrix} = \begin{bmatrix} V_H \\ 0 \end{bmatrix},\ S_J \begin{bmatrix} Q_{J,1} \\ Q_{J,2} \end{bmatrix} = \begin{bmatrix} V_J \\ 0 \end{bmatrix}.$$

The matrix $\tilde{\mathcal{Y}}(\gamma)$ may then be constructed from the CS decompositions of $\begin{bmatrix} S_{H,1} \\ S_{H,2} \end{bmatrix}$ and $\begin{bmatrix} S_{J,1} \\ S_{J,2} \end{bmatrix}$.

A difficulty that could arise here, is that $\begin{bmatrix} Q_{H,1} \\ Q_{H,2} \end{bmatrix}$ and/or $\begin{bmatrix} Q_{J,1} \\ Q_{J,2} \end{bmatrix}$ may be ill-conditioned or may be small norm sections of the matrices with orthonormal columns Q_H and Q_J in (24). Such a problem may be either traced back to an ill-conditioning of the problem of computing the invariant subspace or to a near failure of one or some of assumptions A1-A4. In both cases we cannot expect a solution to be accurate, but clearly then the same or worse problems arise in the reduced pencils such as (26).

If $R_H(\gamma)$ or $R_J(\gamma)$ are nearly singular, then the pencils (22) and (23) are close to pencils that are either not regular or have index greater than one. In this case we are close to a situation, where the dimension of the deflating subspace associated with the open left half plane eigenvalues becomes less than n. If $\hat{\gamma}_H < \gamma_{mo}$ and $\hat{\gamma}_J < \gamma_{mo}$, then this does not happen for $\gamma \geq \gamma_{mo}$. Example 3.1 demonstrates that $\gamma_{mo} = \hat{\gamma}_H$ is possible and the pencil $\lambda N + M_H$ becomes singular near γ_{mo}.

In summary, numerical computations based on the even pencils (22) and (23) avoid unnecessary rounding errors caused by explicitly forming $H(\gamma)$, $J(\gamma)$, and the corresponding algebraic Riccati solutions. Deflating subspaces of the even pencils (22) and (23) provide the desired Lagrangian subspaces, and the factors of $\mathcal{Y}(\gamma)$ and $\tilde{\mathcal{Y}}(\gamma)$ without explicitly forming the inverses, sums and products that occur in (7) and (8).

5 Computation of γ_{mo}

In this section we synthesize the above observations in a new numerical method for the modified optimal H_∞ control problem.

The simplest approach to finding γ_{mo} is to use a bisection method. Given a number $\gamma \geq 0$, the following procedure may be used to determine whether $\gamma \leq \gamma_{mo}$ or $\gamma \geq \gamma_{mo}$.

Algorithm 1 (Basic bisection procedure)

1. *Form the even pencils (22) and (23).*
2. *Use a structure preserving method such as those discussed in [5, 15] to compute the deflating subspaces Q_H and Q_J associated with the eigenvalues in the closed left half plane.*
3. *If the dimension of one or both of these subspaces is less than n, then report $\gamma < \gamma_{mo}$ and STOP.*

4. *Compute the symplectic QR decomposition of the two matrices in (25) followed by the CS decompositions (12)–(15).*

5. *If any diagonal element of $\Delta_H \Sigma_H$ or $\Delta_J \Sigma_J$ is negative, then report $\gamma < \gamma_{mo}$ and STOP.*

6. *Form the matrix $\tilde{\mathcal{Y}}$.*

7. *If $\tilde{\mathcal{Y}}$ is not positive semidefinite, then report $\gamma < \gamma_{mo}$ and STOP.*

8. *If $\tilde{\mathcal{Y}}$ is positive semidefinite and $\operatorname{rank} \tilde{\mathcal{Y}} < \hat{k}_H + \hat{k}_J$ then report $\gamma < \gamma_{mo}$ and STOP ($\hat{k}_H$ and $\hat{k}_J$ can be computed with a sufficiently large γ.)*

9. *Report $\gamma > \gamma_{mo}$.*

Often, γ_{mo} is a root of the function $f(\gamma)$ described in Remark 4.7. Since the eigenvalues of a symmetric matrix are continuous functions of the entries of the matrix (hence also of γ) and continuously differentiable as long as the eigenvalue is simple [47], the secant method applies. We then have the following basic structure of the optimization procedure.

Algorithm 2 (Basic optimization procedure)

1. *Compute upper and lower bounds γ_{up} and γ_{low} for γ_{mo}.*

2. *Use the bisection method (Algorithm 1) to determine a sufficiently small interval $[\gamma_0, \gamma_1]$ in which γ_{mo} lies.*

3. *Use a superlinearly convergent method such as the secant method to determine γ.*

This algorithm needs to fall back upon the bisection procedure in case the secant method produces an approximate root γ for which $\mathcal{Y}(\gamma)$ does not exist.

6 Numerical Examples

In this section we solve several H_∞ control problems and compare our experimental implementation of Algorithm 2 with `Hinfopt` (version 1.8) from the MATLAB Robust Control Toolbox (version 2.0.7) [17]. We used the same highly demanding stopping criterion $\texttt{tolX} = 10^{-14}$ for stopping the γ iteration in both programs. All the numerical examples were run on a Dell 530 workstation using MATLAB (version 6.0.0.88) with IEEE754 conforming floating point arithmetic. The unit round is approximately 2.22×10^{-16}.

Example 6.1 For

$$
\begin{aligned}
A &= \begin{bmatrix} -a & 0 & 1 & -2 & 1 \\ 0 & -100 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2a & a \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 & 2 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1 \\ 0 \\ a \\ 0 \\ 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ -90 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
C_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \; D_{11} = 0, \; D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
C_2 &= \begin{bmatrix} 0 & 0 & 1 & -2 & 1 \end{bmatrix}, \; D_{21} = 1, \; D_{22} = 0
\end{aligned}
$$

γ_{mo} is independent of the choice of a. As is typical, $\hat{\gamma}^{\rho}$ is greater than $\hat{\gamma}$, $\hat{\gamma}^{R}$, $\hat{\gamma}^{I}$ and $\hat{\gamma}^{L}$, so $\gamma_{mo} = \hat{\gamma}^{\rho}$. Our experimental program determined $\gamma_{mo} = \hat{\gamma}^{\rho} = 7.853923684022$ which is correct to roughly thirteen significant digits. This program computed the same optimal value of γ to at least thirteen significant digits for values of a between 1 and 10^{-7}. When $a = 10^{-8}$, then the pencil $\lambda N - M_H$ has finite eigenvalues of magnitude comparable to (and possibly smaller than) the unit round of the floating point arithmetic. At that point, eigenvalue based numerical methods are no longer able to reliably extract the stable deflating subspace. The experimental program delivers an error message. `Hinfopt` gets the same accuracy for a as small as 10^{-10} despite the growing unreliability of the computed eigenvalues as a decreases below 10^{-8}.

Figure 1 shows the nonzero eigenvalues of $\tilde{\mathcal{Y}}(\gamma)$ as a function of γ for $a = 1$. In this example, $\mathcal{Y}(\gamma)$ and $\tilde{\mathcal{Y}}(\gamma)$ have an eigenvalue of magnitude roughly 10^{-6} in the neighborhood of $\hat{\gamma}^{\rho}$, but it is one of the other, relatively larger eigenvalues that changes sign at $\hat{\gamma}^{\rho}$. This example demonstrates that, counter to intuition, a relatively small eigenvalue of $\mathcal{Y}(\gamma)$ or $\tilde{\mathcal{Y}}(\gamma)$ does not necessarily imply that $\gamma \approx \hat{\gamma}^{\rho}$.

Example 6.2 (Example 3.1 continued) In this example $\gamma_{mo} = \hat{\gamma}$. With $\alpha = \beta = \delta = \eta = 1$ and $\epsilon_1 = \epsilon_2 = 0$, the experimental program determined $\gamma_{mo} = \hat{\gamma} = .5000000000000$ which agrees with the theoretical value to thirteen significant digits.

Note that $R_H(\gamma)$ is singular at $\gamma = \gamma_{mo} = \hat{\gamma}$. `Hinfopt` fails on this example, because it explicitly inverts the singular matrix $R_H(\hat{\gamma})$.

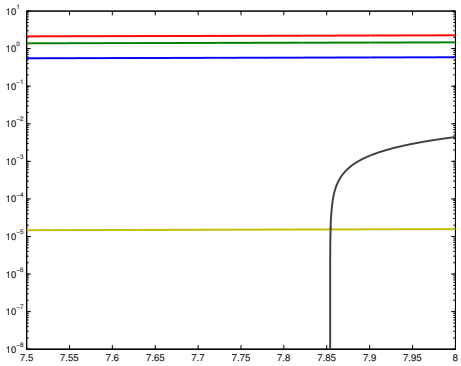
Example 6.3 (Example 3.1 continued) Example 3.1 with $\alpha = \beta = \delta = \eta = \epsilon_2 = 1$ and $\epsilon_1 = 0$ demonstrates a case in which $\gamma_{mo} = \hat{\gamma}^{L}$. As shown in Figure 1, $\hat{\mathcal{Y}}(\gamma)$ does not change rank at $\gamma = \gamma_{mo}$, instead, it ceases to exist, because the semi-stabilizing Lagrangian subspace ceases to exist. The Riccati solution to (8) is $X_J = 0$ independent of γ. The Riccati solution to (7) is not constant, but remains positive definite in a one sided neighborhood to the right of γ_{mo}. In a neighborhood to the left of γ_{mo}, the Hamiltonian matrix $H(\gamma)$ (7) and the pencil $\lambda N - M_H$ have eigenvalues with zero real part and the required Lagrangian invariant subspaces fail to exist. Our experimental code reports $\gamma_{mo} = \hat{\gamma}^{L} = .8062257748299$. `Hinfopt` fails on this example, because it explicitly inverts the singular matrix $R_H(\gamma)$.

Example 6.4 In this example the H_∞ norm of T_{zw} is nearly minimized by a large range of values γ using the γ-parameterization of Theorem 2.7, including a region below γ_{mo}. That is, using any of these γ's to construct a controller, nearly the same H_∞ norm of T_{zw} is attained. Let

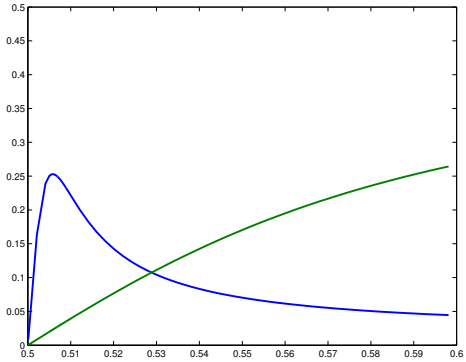
$$\left[\begin{array}{c|c|c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{21} & 0 \end{array}\right] = \left[\begin{array}{cc|cc|c} 2 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 & -2 \\ \hline 1 & 0 & \alpha & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ \hline 4 & -2 & 0 & 1 & 0 \end{array}\right].$$

Then $\hat{\gamma} = \gamma_{mo} = \alpha$. Taking $\alpha = 3$ one can verify that, except for $\gamma \in [2.7, 3]$, the Lagrangian subspaces and Riccati solutions exist. But note that for $\gamma < 3$, Condition 1. of Theorem 2.7 is not satisfied, so $\|T_{zw}\|_\infty < 3$ cannot be achieved. Using the formulas in [54] we constructed a controller for each $\gamma \in [1.5, 4] \setminus [2.7, 3]$ and found that $\|T_{zw}\|_\infty = 3.00$ to three significant digits independent of γ.

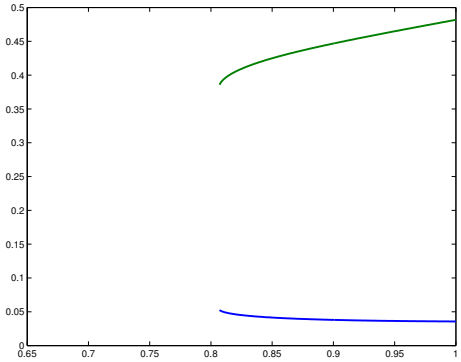
Figure 1 shows the nonzero eigenvalues of $\tilde{\mathcal{Y}}(\gamma)$ for $\gamma \in [.5, 3.5]$. The Riccati solutions X_H of (7) and X_J of (8) have the peculiar property that $X_J(\gamma) \equiv 0$ and $\lim_{\gamma \to \gamma_{mo}+} X_H(\gamma) = 0$, so



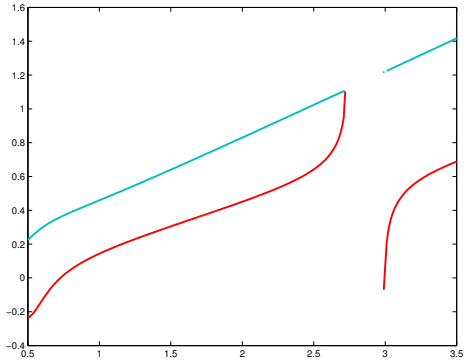
Example 6.1 with $a = 1$: $\gamma_{mo} = \hat{\gamma}^{\rho} \approx 7.853923684022$.



Example 6.2: $\gamma_{mo} = \hat{\gamma} = .5$.



Example 6.3: $\gamma_{mo} = \hat{\gamma}^{L} = .806\ldots$.



Example 6.4: $\gamma_{mo} = \hat{\gamma} = 3$.

Figure 1: Nonzero eigenvalues of $\tilde{\mathcal{Y}}(\gamma)$ from Examples 6.1, 6.2, 6.3 and 6.4 as a function of γ. Graphs of the eigenvalues of $\mathcal{Y}(\gamma)$ are similar.

$\rho(X_H X_J) = 0$ independent of γ. When $\gamma \approx \gamma_{mo}$, a small error in X_J may lead to a relatively large error in the computed spectral radius $\rho(X_J X_H)$. An inaccurately computed spectral radius may limit the accuracy attainable by conventional algorithms that rely on Theorem 2.7 and explicit calculation of Riccati solutions. Nevertheless, `Hinfopt` correctly determined γ_{mo} to within an absolute error of 10^{-13} as did our experimental algorithm described in this paper.

7 Conclusion

This paper discusses the design of a robust numerical method for the modified H_∞ control problem. The proposed method avoids matrix sums, products and inverses needed to construct Hamiltonian matrices and avoids potentially ill-conditioned algebraic Riccati equations by working with even pencils and its deflating subspaces. The computation of the optimal γ reduces to a one-dimensional optimization problem for which, in principle, one can apply quadratically convergent methods. Several examples illustrate the numerical hazards and the

properties of the proposed numerical method. The new approach effectively increases the set of problems to which H_∞ control may be applied.

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