Singular-value-like decomposition for complex matrix triples

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December 17, 2007

Dedicated to William B. Gragg on the occasion of his 70th birthday

Abstract

The classical singular value decomposition for a matrix $A \in \mathbb{C}^{m\times n}$ is a canonical form for A that also displays the eigenvalues of the Hermitian matrices AA^* and A^*A. In this paper, we develop a corresponding decomposition for A that provides the Jordan canonical forms for the complex symmetric matrices AA^T and A^TA. More generally, we consider the matrix triple $(A, G, \hat{G})$, where $G \in \mathbb{C}^{m\times m}, \hat{G} \in \mathbb{C}^{n\times n}$ are invertible and either complex symmetric or complex skew-symmetric, and we provide a canonical form under transformations of the form $(A, G, \hat{G}) \mapsto (X^TAY, X^TGX, Y^T\hat{G}Y)$, where X, Y are nonsingular.

Keywords singular value decomposition, canonical form, complex bilinear form, complex symmetric matrix, complex skew-symmetric matrix, Hamiltonian matrix, Takagi factorization.

AMS subject classification. 65F15, 65L80, 65L05, 15A21, 34A30, 93B40.

1 Introduction

In [3] Bunse-Gerstner and Gragg derived an algorithm for computing the Takagi factorization $A = U^T\Sigma U$, U unitary, for a complex symmetric matrix $A^T = A \in \mathbb{C}^{n\times n}$. The Takagi factorization is just a special case of the singular value decomposition and combines two important aspects: computation of singular values (i.e., eigenvalues of A^*A and AA^*) and exploitation of structure with respect to complex bilinear forms (here, the symmetry of A is exploited by choosing U and U^T as unitary factors for the singular value decomposition).

These two aspects can be combined in a completely different way. Instead of computing the singular values of a general matrix $A \in \mathbb{C}^{m\times n}$ and thus revealing the eigenvalues of AA^* and A^*A, we may ask for a canonical form for A that reveals the eigenvalues of the complex

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Partially supported by *Senior Visiting Scholar Fund of Fudan University Key Laboratory* and *the University of Kansas General Research Fund allocation # 2301717. Part of the work was done while this author was visiting Fudan University and TU Berlin whose hospitality is gratefully acknowledged.*

[*]Partially supported by the *Deutsche Forschungsgemeinschaft* through the DFG Research Center MATHEON *Mathematics for key technologies* in Berlin.

symmetric matrices AA^T and A^TA. In this paper, we compute such a form by solving a more general problem: instead of restricting ourselves to the matrix A, we consider a triple of matrices $(A, G, \hat{G})$ with $A \in \mathbb{C}^{m\times n}$, $G \in \mathbb{C}^{m\times m}$ and $\hat{G} \in \mathbb{C}^{n\times n}$, where G and $\hat{G}$ are nonsingular and either complex symmetric or complex skew-symmetric. Then we derive canonical forms under transformations of the form

$$(A, G, \hat{G}) \mapsto (A_{\mathrm{CF}}, G_{\mathrm{CF}}, \hat{G}_{\mathrm{CF}}) := (X^TAY, X^TGX, Y^T\hat{G}Y), \tag{1.1}$$

with nonsingular matrices $X \in \mathbb{C}^{m\times m}$ and $Y \in \mathbb{C}^{n\times n}$. This canonical form will allow the determination of the eigenstructure of the pair of structured matrices

$$\hat{\mathcal{H}} = \hat{G}^{-1}A^TG^{-1}A, \qquad \mathcal{H} = G^{-1}A\hat{G}^{-1}A^T,$$

because we find that

$$Y^{-1}\hat{\mathcal{H}}Y = (Y^{-1}\hat{G}^{-1}Y^{-T})(Y^TA^TX)(X^{-1}G^{-1}X^{-T})(X^TAY) = \hat{G}_{\mathrm{CF}}^{-1}A_{\mathrm{CF}}^TG_{\mathrm{CF}}^{-1}A_{\mathrm{CF}}, \tag{1.2}$$

$$X^{-1}\mathcal{H}X = (X^{-1}G^{-1}X^{-T})(X^TAY)(Y^{-1}\hat{G}^{-1}Y^{-T})(Y^TA^TX) = G_{\mathrm{CF}}^{-1}A_{\mathrm{CF}}\hat{G}_{\mathrm{CF}}^{-1}A_{\mathrm{CF}}^T. \tag{1.3}$$

For the special case $G = I_m$ and $\hat{G} = I_n$, we obtain $\hat{\mathcal{H}} = A^TA$ and $\mathcal{H} = AA^T$ and thus, an appropriate canonical form (1.1) will display the eigenvalues of A^TA and AA^T via the identities (1.2) and (1.3). In the general case, if $G^T = (-1)^sG$ and $\hat{G}^T = (-1)^t\hat{G}$ with $s, t \in \{0, 1\}$, then the matrices $\hat{\mathcal{H}}$ and $\mathcal{H}$ satisfy

$$\hat{\mathcal{H}}^T\hat{G} = (-1)^sA^TG^{-1}A = (-1)^s\hat{G}\hat{\mathcal{H}}, \quad \mathcal{H}^TG = (-1)^tA\hat{G}^{-1}A^T = (-1)^tG\mathcal{H}, \tag{1.4}$$

i.e., $\hat{\mathcal{H}}$ and $\mathcal{H}$ are either selfadjoint or skew-adjoint with respect to the complex bilinear form induced by $\hat{G}$ or G, respectively. Indeed, setting

$$\langle x, y\rangle_G = y^TGx, \qquad \langle x, y\rangle_{\hat{G}} = y^T\hat{G}x \tag{1.5}$$

for $x, y \in \mathbb{C}^n$, the identities (1.4) can be rewritten as

$$\langle\hat{\mathcal{H}}x, y\rangle_{\hat{G}} = (-1)^s\langle x, \hat{\mathcal{H}}y\rangle_{\hat{G}} \quad \text{and} \quad \langle\mathcal{H}x, y\rangle_G = (-1)^t\langle x, \mathcal{H}y\rangle_G \quad \text{for all } x, y \in \mathbb{C}^n.$$

Indefinite inner products and related structured matrices have been intensively studied in the last few decades with main focus on real bilinear or complex sesquilinear forms, see [1, 5, 12, 15] and the references therein and, in particular, [6]. In recent years, there has also been interest in matrices that are structured with respect to complex bilinear forms, because such matrices do appear in applications such as the frequency analysis of high speed trains [8, 13].

Besides revealing the eigenstructure of the matrices $\hat{\mathcal{H}}$ and $\mathcal{H}$, the canonical form (1.1) also allows to determine the eigenstructure of the double-sized structured matrix pencil

$$\lambda\mathcal{G} - \mathcal{A} = \lambda\begin{bmatrix} G & 0 \\ 0 & \hat{G} \end{bmatrix} - \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix},$$

because we have that

$$\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}^T \left(\lambda\begin{bmatrix} G & 0 \\ 0 & \hat{G} \end{bmatrix} - \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}\right)\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$$
$$= \lambda\begin{bmatrix} G_{\mathrm{CF}} & 0 \\ 0 & \hat{G}_{\mathrm{CF}} \end{bmatrix} - \begin{bmatrix} 0 & A_{\mathrm{CF}} \\ A_{\mathrm{CF}}^T & 0 \end{bmatrix}.$$

The idea of generalizing the concept of the singular value decomposition to indefinite inner products by considering transformations of the form (1.1) is not new and has been considered in [2] for the case of complex Hermitian forms. The canonical forms presented here are the analogue in the case of complex bilinear forms. This case is more involved, because one has to make a clear distinction between symmetric and skew-symmetric bilinear forms, in contrast to the sesquilinear case, where Hermitian and skew-Hermitian forms are closely related. Indeed, an Hermitian matrix can be easily transformed into a skew-Hermitian matrix by scalar multiplication with the imaginary unit i, but this is not true for complex symmetric matrices. Therefore, we have to treat the three cases separately that G and $\hat{G}$ are both symmetric, both skew-symmetric, or that one of the matrices is symmetric and the another skew-symmetric.

A canonical form closely related to the form obtained under the transformation (1.1) has been developed in [11], where transformations of the form

$$(B, C) \mapsto (X^{-1}BY, Y^{-1}CX), \quad B \in \mathbb{C}^{m \times n}, C \in \mathbb{C}^{n \times m}$$

have been considered. Then a canonical form is constructed that reveals the Jordan structures of the products BC and CB. In our framework, this corresponds to a canonical form of the pair of matrices $(G^{-1}A, \hat{G}^{-1}A^T)$ rather than for the triple $(A, G, \hat{G})$. In this case our approach is more general, because the canonical form for the pair $(G^{-1}A, \hat{G}^{-1}A^T)$ can be easily read off the canonical form for $(A, G, \hat{G})$, but not vice versa. The approach in [11], on the other hand, focusses on different aspects and allows to consider pairs (B, C) where the ranks of B and C are distinct. This situation is not covered by the canonical forms obtained in this paper.

The remainder of the paper is organized as follows. In Section 2 we recall the definition of several structured matrices and review their canonical forms. In Section 3 we develop structured factorizations that are needed for the proofs of the results in the following sections. In Sections 4–6 we present the canonical forms for matrix triples $(A, G, \hat{G})$. In Section 4 we consider the case that both G and $\hat{G}$ are complex symmetric, in Section 5 we assume that G is complex symmetric and $\hat{G}$ is complex skew-symmetric, and Section 6 is devoted to the case that both G and $\hat{G}$ are complex skew-symmetric.

Throughout the paper we use the following notation. I_n and 0_n denote the $n \times n$ identity and $n \times n$ zero matrices, respectively. The $m \times n$ zero matrix is denoted by $0_{m \times n}$ and e_j is the jth column of the identity matrix I_n, or, equivalently, the jth standard basis vector of $\mathbb{C}^n$. Moreover, we denote

$$R_n := \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}, \ \Sigma_{m,n} := \begin{bmatrix} I_m & 0 \\ 0 & -I_n \end{bmatrix}, \ J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \ \mathcal{J}_n(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}.$$

The transpose and conjugate transpose of a matrix A are denoted by A^T and A^*, respectively. We use $A_1 \oplus \cdots \oplus A_k$ to denote a block diagonal matrix with diagonal blocks $A_1, \ldots, A_k$. If $A = [a_{ij}] \in \mathbb{C}^{n \times m}$ and $B \in \mathbb{C}^{\ell \times k}$, then $A \otimes B = [a_{ij}B] \in \mathbb{C}^{n\ell \times mk}$ denotes the Kronecker product of A and B.

2 Matrices structured with respect to complex bilinear forms

Our general theory will cover and generalize results for the following classes of matrices.

Definition 2.1 *Let $G \in \mathbb{C}^{n\times n}$ be invertible and let $\mathcal{H}, \mathcal{K} \in \mathbb{C}^{n\times n}$ such that*

$$(G\mathcal{H})^T = G\mathcal{H} \quad \text{and} \quad (G\mathcal{K})^T = -G\mathcal{K}.$$

1. *If G is symmetric, then $\mathcal{H}$ is called G-*symmetric *and $\mathcal{K}$ is called G-*skew-symmetric.
2. *If G is skew-symmetric, then $\mathcal{H}$ is called G-*Hamiltonian *and $\mathcal{K}$ is called G-*skew-Hamiltonian.

Thus, G-symmetric and G-skew-Hamiltonian matrices are selfadjoint in the inner product induced by G, while G-skew-symmetric and G-Hamiltonian matrices are skew-adjoint. Observe that transformations of the form

$$(\mathcal{M}, G) \mapsto (P^{-1}\mathcal{M}P, P^TGP), \quad P \in \mathbb{C}^{n\times n} \text{ invertible}$$

preserve the structure of $\mathcal{M}$ with respect to G, i.e., if, for example, $\mathcal{M} = \mathcal{H}$ is G-Hamiltonian, then $P^{-1}\mathcal{H}P$ is P^TGP-Hamiltonian as well. Thus, instead of working with G directly, one may first transform G to a simple form using the Takagi factorization for complex symmetric and complex skew-symmetric matrices, see [3, 9, 16]. This factorization is a special case of the well-known singular value decomposition.

Theorem 2.2 (Takagi's factorization) *Let $G \in \mathbb{C}^{n\times n}$ be complex symmetric. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$G = U\text{diag}(\sigma_1, \ldots, \sigma_n)U^T, \quad \text{where } \sigma_1, \ldots, \sigma_n \geq 0.$$

There is a variant for complex skew-symmetric matrices (see [9]). This result is a just a special case of the Youla form [18] for general complex matrices.

Theorem 2.3 (Skew-symmetric analogue of Takagi's factorization) *Let $\mathcal{K} \in \mathbb{C}^{n\times n}$ be complex skew-symmetric. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that*

$$\mathcal{K} = U\left(\begin{bmatrix} 0 & r_1 \\ -r_1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & r_k \\ -r_k & 0 \end{bmatrix} \oplus 0_{n-2k}\right)U^T,$$

where $r_1, \ldots, r_n \in \mathbb{R} \setminus \{0\}$.

As immediate corollaries, we obtain the following well-known results.

Corollary 2.4 *Let $G \in \mathbb{C}^{n\times n}$ be complex symmetric and let* $\operatorname{rank} G = r$*. Then there exists a nonsingular matrix $X \in \mathbb{C}^{n\times n}$ such that*

$$X^TGX = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

Corollary 2.5 *Let $G \in \mathbb{C}^{m\times m}$ be complex skew-symmetric and let* $\operatorname{rank} G = r$*. Then r is even and there exists a nonsingular matrix $X \in \mathbb{C}^{n\times n}$ such that*

$$X^TGX = \begin{bmatrix} J_{r/2} & 0 \\ 0 & 0 \end{bmatrix}.$$

Next, we review canonical forms for the classes of matrices defined in Definition 2.1. These canonical forms are closely related to the well-known canonical forms for pairs of matrices that are complex symmetric or complex skew-symmetric, see [17] for an overview on this topic. Proofs of the following results can be found, e.g., in [14].

Theorem 2.6 (Canonical form for G-symmetric matrices) *Let $G \in \mathbb{C}^{n\times n}$ be symmetric and invertible and let $\mathcal{H} \in \mathbb{C}^{n\times n}$ be G-symmetric. Then there exists an invertible matrix $X \in \mathbb{C}^{n\times n}$ such that*

$$X^{-1}\mathcal{H}X = \mathcal{J}_{\xi_1}(\lambda_1) \oplus \ldots \oplus \mathcal{J}_{\xi_m}(\lambda_m), \quad X^T G X = R_{\xi_1} \oplus \ldots \oplus R_{\xi_m},$$

where $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ are the (not necessarily pairwise distinct) eigenvalues of $\mathcal{H}$.

For the next two results, we need additional notation. By Γ_η we denote the matrix with alternating signs on the anti-diagonal, i.e.,

$$\Gamma_\eta = \begin{bmatrix} 0 & & & (-1)^0 \\ & & (-1)^1 & \\ & \iddots & & \\ (-1)^{\eta-1} & & & 0 \end{bmatrix}.$$

Theorem 2.7 (Canonical form for G-skew-symmetric matrices) *Let $G \in \mathbb{C}^{n\times n}$ be symmetric and invertible and let $\mathcal{K} \in \mathbb{C}^{n\times n}$ be G-skew-symmetric. Then there exists an invertible matrix $X \in \mathbb{C}^{n\times n}$ such that*

$$X^{-1}\mathcal{K}X = \mathcal{K}_c \oplus \mathcal{K}_z, \quad X^T G X = G_c \oplus G_z,$$

where

$$\begin{array}{ll} \mathcal{K}_c = \mathcal{K}_{c,1} \oplus \cdots \oplus \mathcal{K}_{c,m_c}, & G_c = G_{c,1} \oplus \cdots \oplus G_{c,m_c}, \\ \mathcal{K}_z = \mathcal{K}_{z,1} \oplus \cdots \oplus \mathcal{K}_{z,m_o+m_e}, & G_z = G_{z,1} \oplus \cdots \oplus G_{z,m_o+m_e}, \end{array}$$

and where the diagonal blocks are given as follows:

1) *blocks associated with pairs $(\lambda_j, -\lambda_j)$ of nonzero eigenvalues of $\mathcal{K}$:*

$$\mathcal{K}_{c,j} = \begin{bmatrix} \mathcal{J}_{\xi_j}(\lambda_j) & 0 \\ 0 & -\mathcal{J}_{\xi_j}(\lambda_j) \end{bmatrix}, \quad G_{c,j} = \begin{bmatrix} 0 & R_{\xi_j} \\ R_{\xi_j} & 0 \end{bmatrix},$$

   where $\lambda_j \in \mathbb{C} \setminus \{0\}$ and $\xi_j \in \mathbb{N}$ for $j = 1, \ldots, m_c$ when $m_c > 0$;

2) *blocks associated with the eigenvalue $\lambda = 0$ of $\mathcal{K}$:*

$$\mathcal{K}_{z,j} = \mathcal{J}_{\eta_j}(0), \quad G_{z,j} = \Gamma_{\eta_j},$$

   where $\eta_j \in \mathbb{N}$ is odd for $j = 1, \ldots, m_o$ when $m_o > 0$, and

$$\mathcal{K}_{z,j} = \begin{bmatrix} \mathcal{J}_{\eta_j}(0) & 0 \\ 0 & -\mathcal{J}_{\eta_j}(0) \end{bmatrix}, \quad G_{z,j} = \begin{bmatrix} 0 & R_{\eta_j} \\ R_{\eta_j} & 0 \end{bmatrix},$$

   where $\eta_j \in \mathbb{N}$ is even for $j = m_o + 1, \ldots, m_o + m_e$ when $m_e > 0$.

The matrix $\mathcal{K}$ has the non-zero eigenvalues $\lambda_1, \ldots, \lambda_{m_c}, -\lambda_1, \ldots, -\lambda_{m_c}$ (not necessarily pairwise distinct), and the additional eigenvalue 0 if $m_o + m_e > 0$.

Theorem 2.8 (Canonical form for G-Hamiltonian matrices) *Let $G \in \mathbb{C}^{2n\times 2n}$ be complex skew-symmetric and invertible and let $\mathcal{H} \in \mathbb{C}^{2n\times 2n}$ be G-Hamiltonian. Then there exists an invertible matrix $X \in \mathbb{C}^{2n\times 2n}$ such that*

$$X^{-1}\mathcal{H}X = \mathcal{H}_c \oplus \mathcal{H}_z, \quad X^T G X = G_c \oplus G_z,$$

where

$$\begin{array}{ll} \mathcal{H}_c = \mathcal{H}_{c,1} \oplus \cdots \oplus \mathcal{H}_{c,m_c}, & G_c = G_{c,1} \oplus \cdots \oplus G_{c,m_c}, \\ \mathcal{H}_z = \mathcal{H}_{z,1} \oplus \cdots \oplus \mathcal{H}_{z,m_o+m_e}, & G_z = G_{z,1} \oplus \cdots \oplus G_{z,m_o+m_e}, \end{array}$$

and where the diagonal blocks are given as follows:

1) *blocks associated with pairs $(\lambda_j, -\lambda_j)$ of nonzero eigenvalues of $\mathcal{H}$:*

$$\mathcal{H}_{c,j} = \begin{bmatrix} \mathcal{J}_{\xi_j}(\lambda_j) & 0 \\ 0 & -\mathcal{J}_{\xi_j}(\lambda_j) \end{bmatrix}, \quad G_{c,j} = \begin{bmatrix} 0 & R_{\xi_j} \\ -R_{\xi_j} & 0 \end{bmatrix},$$

where $\lambda_j \in \mathbb{C} \setminus \{0\}$ with $\arg(\lambda_j) \in [0, \pi)$ and $\xi_j \in \mathbb{N}$ for $j = 1, \ldots, m_c$ when $m_c > 0$;

2) *blocks associated with the eigenvalue $\lambda = 0$ of $\mathcal{H}$:*

$$\mathcal{H}_{z,j} = \begin{bmatrix} \mathcal{J}_{\xi_j}(0) & 0 \\ 0 & -\mathcal{J}_{\xi_j}(0) \end{bmatrix}, \quad G_{z,j} = \begin{bmatrix} 0 & R_{\xi_j} \\ -R_{\xi_j} & 0 \end{bmatrix},$$

where $\eta_j \in \mathbb{N}$ is odd for $j = 1, \ldots, m_o$ when $m_o > 0$, and

$$\mathcal{H}_{z,j} = \mathcal{J}_{\eta_j}(0), \quad G_{z,j} = \mathit{\Gamma}_{\eta_j}$$

where $\eta_j \in \mathbb{N}$ is even for $j = m_o + 1, \ldots, m_o + m_e$ when $m_e > 0$.

The matrix $\mathcal{H}$ has the non-zero eigenvalues $\lambda_1, \ldots, \lambda_{m_c}, -\lambda_1, \ldots, -\lambda_{m_c}$ (not necessarily pairwise distinct), and the additional eigenvalue 0 if $m_o + m_e > 0$.

Theorem 2.9 (Canonical form for G-skew-Hamiltonian matrices) *Let $G \in \mathbb{C}^{2n\times 2n}$ be complex skew-symmetric and invertible and let $\mathcal{K} \in \mathbb{C}^{2n\times 2n}$ be G-skew-Hamiltonian. Then there exists an invertible matrix $X \in \mathbb{C}^{2n\times 2n}$ such that*

$$X^{-1}\mathcal{K}X = \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_m, \quad X^T G X = G \oplus \cdots \oplus G_m,$$

where

$$\mathcal{K}_j = \begin{bmatrix} \mathcal{J}_{\xi_j}(\lambda_j) & 0 \\ 0 & \mathcal{J}_{\xi_j}(\lambda_j) \end{bmatrix}, \quad G_j = \begin{bmatrix} 0 & R_{\xi_j} \\ -R_{\xi_j} & 0 \end{bmatrix}.$$

The matrix $\mathcal{K}$ has the (not necessarily pairwise distinct) eigenvalues $\lambda_1, \ldots, \lambda_m$.

The following lemma on existence and uniqueness of structured square roots of structured matrices will frequently be used.

Lemma 2.10 *Let $G \in \mathbb{C}^{n\times n}$ be invertible and let $\mathcal{H} \in \mathbb{C}^{n\times n}$ be invertible and such that $\mathcal{H}^T G = G\mathcal{H}$.*

1. *If $G \in \mathbb{C}^{n\times n}$ is complex symmetric (i.e., $\mathcal{H} \in \mathbb{C}^{n\times n}$ is G-symmetric), then there exists a square root $S \in \mathbb{C}^{n\times n}$ of $\mathcal{H}$ that is a polynomial in $\mathcal{H}$ and that satisfies $\sigma(S) \subseteq \{z \in \mathbb{C} : \arg(z) \in [0, \pi)\}$. The square root is uniquely determined by these properties. In particular, S is G-symmetric.*

2. *If $G \in \mathbb{C}^{n\times n}$ if complex skew-symmetric (i.e., $\mathcal{H} \in \mathbb{C}^{n\times n}$ is G-skew-Hamiltonian), then there exists a square root $S \in \mathbb{C}^{n\times n}$ of $\mathcal{H}$ that is a polynomial in $\mathcal{H}$ and that satisfies $\sigma(S) \subseteq \{z \in \mathbb{C} : \arg(z) \in [0,\pi)\}$. The square root is uniquely determined by these properties. In particular, S is G-skew-Hamiltonian.*

Proof. By the discussion in Chapter 6.4 in [10], we obtain for both cases that a square root S of $\mathcal{H}$ with $\sigma(S) \subseteq \{z \in \mathbb{C} : \arg(z) \in [0,\pi)\}$ exists, is unique, and can be expressed as a polynomial in $\mathcal{H}$. It is straightforward to check that a matrix that is a polynomial in $\mathcal{H}$ is again G-symmetric or G-skew-Hamiltonian, respectively. □

3 Structured factorizations

In this section, we develop basic factorizations that will be needed for computing the canonical forms in the Sections 4–6. We start with factorizations for matrices $B \in \mathbb{C}^{m\times n}$ satisfying $B^TB = I$ or $B^TB = 0$.

Lemma 3.1 *If $B \in \mathbb{C}^{m\times n}$ satisfies $B^TB = I_n$, then $m \geq n$ and there exists a nonsingular matrix $X \in \mathbb{C}^{m\times m}$ such that*

$$X^TB = \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \quad X^TX = I_m.$$

Proof. By assumption B has full column rank. So there exists $\widetilde{B} \in \mathbb{C}^{m\times(m-n)}$ such that $\widetilde{X} = \begin{bmatrix} B & \widetilde{B} \end{bmatrix} \in \mathbb{C}^{m\times m}$ is invertible. Then

$$\widetilde{X}^T\widetilde{X} = \begin{bmatrix} I_n & B^T\widetilde{B} \\ \widetilde{B}^TB & \widetilde{B}^T\widetilde{B} \end{bmatrix},$$

and with

$$X_1 = \begin{bmatrix} I_n & -B^T\widetilde{B} \\ 0 & I_{m-n} \end{bmatrix},$$

we have

$$(\widetilde{X}X_1)^T(\widetilde{X}X_1) = \begin{bmatrix} I_n & 0 \\ 0 & \widetilde{B}^T(I - BB^T)\widetilde{B} \end{bmatrix}.$$

Since $\widetilde{X}X_1$ is nonsingular, so is the complex symmetric matrix $\widetilde{B}^T(I - BB^T)\widetilde{B}$. By Corollary 2.4, there exists a nonsingular matrix X_2 such that

$$X_2^T\big(\widetilde{B}^T(I - BB^T)\widetilde{B}\big)X_2 = I_{m-n}.$$

With

$$X = \widetilde{X}X_1 \begin{bmatrix} I_n & 0 \\ 0 & X_2 \end{bmatrix}$$

we then obtain $X^TX = I_m$. Note that

$$X = \begin{bmatrix} B & \widetilde{B} \end{bmatrix} \begin{bmatrix} I_n & -B^T\widetilde{B} \\ 0 & I_{m-n} \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & X_2 \end{bmatrix} = \begin{bmatrix} B & (I - BB^T)\widetilde{B}X_2 \end{bmatrix},$$

and hence $X^TX = I_m$ implies that

$$X^TB = \begin{bmatrix} I_n \\ 0 \end{bmatrix}. \quad \square$$

Lemma 3.2 *If $B \in \mathbb{C}^{m\times n}$ satisfies* $\operatorname{rank} B = n$ *and $B^T B = 0$, then $m \geq 2n$ and there exists a unitary matrix $X \in \mathbb{C}^{m\times m}$ such that*

$$X^T B = \begin{bmatrix} B_0 \\ 0_n \\ 0 \end{bmatrix}, \quad X^T X = \begin{bmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_{m-2n} \end{bmatrix},$$

where $B_0 \in \mathbb{C}^{n\times n}$ is upper triangular and invertible.

Proof. We present a constructive proof which allows to determine the matrix X numerically. We may assume that $m \geq 2$, otherwise the result holds trivially. Let

$$Be_1 = u_1 + iv_1, \quad u_1, v_1 \in \mathbb{R}^m.$$

Then (using e.g. a Householder transformation, see [7]) there exists an orthogonal matrix $Q_1 \in \mathbb{R}^{m\times m}$ such that $Q_1^T u_1 = \alpha_1 e_1$ and $0 \leq \alpha_1 \in \mathbb{R}$. Let $\widetilde{v}_1$ be the vector formed by the trailing $m-1$ components of $Q_1^T v_1$. Then (using e.g. a QR-decomposition, see [7]) there exists an orthogonal matrix $Q_2 \in \mathbb{R}^{(m-1)\times(m-1)}$ such that $Q_2^T \widetilde{v}_1 = \beta_1$ and $0 \leq \beta_1 \in \mathbb{R}$. With $U_1 = Q_1(1 \oplus Q_2)$, then

$$U_1^T B = \begin{bmatrix} \alpha_1 + iv_{11} & b_1 \\ i\beta_1 & b_2 \\ 0 & B_1 \end{bmatrix},$$

where $B_1 \in \mathbb{C}^{(m-2)\times(n-1)}$, $b_1, b_2 \in \mathbb{C}^{1\times(n-1)}$, and $v_{11} \in \mathbb{R}$. Since U_1 is real orthogonal, we have

$$(U_1^T B)^T (U_1^T B) = B^T B = 0,$$

and hence,

$$(\alpha_1 + iv_{11})^2 - \beta_1^2 = 0, \quad (\alpha_1 + iv_{11})b_1 + i\beta_1 b_2 = 0, \quad B_1^T B_1 + b_1^T b_1 + b_2^T b_2 = 0_{n-2}. \tag{3.1}$$

From the first identity in (3.1), it follows that $v_{11} = 0$ and $\alpha_1 = \beta_1$. Since $\alpha_1, \beta_1 \geq 0$, we have that $\alpha_1 = \beta_1 > 0$, because otherwise we would have that $\operatorname{rank} B \leq n-1$, which is a contradiction. With this, the last two identities in (3.1) imply that $b_1 = -ib_2$, $B_1^T B_1 = 0$, and thus,

$$U_1^T B = \begin{bmatrix} \alpha_1 & -ib_2 \\ i\alpha_1 & b_2 \\ 0 & B_1 \end{bmatrix}, \qquad B_1 \in \mathbb{C}^{(m-2)\times(n-1)}.$$

One can easily verify that $\operatorname{rank} B_1 = n-1$.

Applying the same procedure inductively to B_1 we obtain the existence of a real orthogonal matrix U_2 such that

$$U_2^T B_1 = \begin{bmatrix} \alpha_2 & -ib_3 \\ i\alpha_2 & b_3 \\ 0 & B_2 \end{bmatrix}, \qquad B_2 \in \mathbb{C}^{(m-4)\times(n-2)}.$$

Similarly as above, we can show that $\alpha_2 > 0$ and $\operatorname{rank} B_2 = n-2$.

Continuing the procedure, we finally obtain a real orthogonal matrix U such that

$$UB = \begin{bmatrix} \alpha_1 & -ib_{12} & \dots & -ib_{1n} \\ i\alpha_1 & b_{12} & \dots & b_{1n} \\ & \alpha_2 & \dots & -ib_{2n} \\ & i\alpha_2 & \dots & b_{2n} \\ & & \ddots & \vdots \\ & & & \alpha_n \\ & & & -i\alpha_n \\ & & & 0 \\ & & & \vdots \\ & & & 0 \end{bmatrix}.$$

and from this we obtain that $m \geq 2n$. Moreover, we see that every other row of UB is a multiple by i of the preceding row. Thus, setting

$$Z_1 = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}, \qquad Z = \underbrace{Z_1 \oplus \ldots \oplus Z_1}_{n} \oplus I_{m-2n},$$

letting P be a permutation matrix for which premultiplication has the effect of re-arranging the first $2n$ rows of a matrix in the order of $1, 3, \ldots, 2n-1, 2, 4, \ldots, 2n$, and introducing the unitary matrix $X = (PZU)^T$, we then have

$$X^T B = \sqrt{2} \begin{bmatrix} \alpha_1 & -ib_{12} & \dots & -ib_{1n} \\ & \alpha_2 & \dots & -ib_{2n} \\ & & \ddots & \vdots \\ & & & \alpha_n \\ & & & 0 \\ & & & \vdots \\ & & & 0 \end{bmatrix}.$$

and we obtain furthermore that

$$ZZ^T = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{n} \oplus I_{m-2n} \quad \text{and} \quad X^T X = \begin{bmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_{m-2n} \end{bmatrix},$$

using the fact that U is real orthogonal, i.e., $U^T U = I$. □

Proposition 3.3 *Let $B \in \mathbb{C}^{m \times n}$ and suppose that $\operatorname{rank} B = n$, $\operatorname{rank} B^T B = n_0 \leq n$, and that $\delta_0 = n - n_0$ is the dimension of the null space of $B^T B$. Then there exists a nonsingular $X \in \mathbb{C}^{m \times m}$ such that*

$$X^T B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix} \begin{matrix} m-n \\ n \end{matrix}, \quad X^T X = I_{n_1} \oplus \begin{bmatrix} 0 & 0 & I_{\delta_0} \\ 0 & I_{n_0} & 0 \\ I_{\delta_0} & 0 & 0 \end{bmatrix},$$

where $B_0 \in \mathbb{C}^{n \times n}$ is nonsingular and $n_1 = m - n - \delta_0$.

Proof. Since B^TB is complex symmetric, by the assumption and by Corollary 2.4, there exists a nonsingular matrix $Y \in \mathbb{C}^{n\times n}$ such that

$$Y^TB^TBY = \begin{bmatrix} I_{n_0} & 0 \\ 0 & 0_{\delta_0} \end{bmatrix}.$$

Let $\widetilde{B} \in \mathbb{C}^{m\times n_0}$ be the matrix formed by the leading n_0 columns of BY. By Lemma 3.1 there exists $X_1 \in \mathbb{C}^{m\times m}$ such that

$$X_1^T\widetilde{B} = \begin{bmatrix} I_{n_0} \\ 0 \end{bmatrix}, \quad X_1^TX_1 = I_m$$

and we obtain that

$$X_1^TBY = \begin{bmatrix} I_{n_0} & B_{12} \\ 0 & B_1 \end{bmatrix}.$$

Since

$$(X_1^TBY)^T(X_1^TBY) = Y^TB^TBY = \begin{bmatrix} I_{n_0} & 0 \\ 0 & 0_{\delta_0} \end{bmatrix},$$

we have that

$$B_{12} = 0, \quad B_1^TB_1 = 0_{\delta_0}.$$

By assumption, B has full column rank, so this also holds for $B_1 \in \mathbb{C}^{(m-n_0)\times\delta_0}$. By Lemma 3.2 there exists a nonsingular matrix $X_2 \in \mathbb{C}^{(m-n_0)\times(m-n_0)}$ such that

$$X_2^TB_1 = \begin{bmatrix} T \\ 0_{\delta_0} \\ 0 \end{bmatrix}, \quad X_2^TX_2 = \begin{bmatrix} 0 & I_{\delta_0} & 0 \\ I_{\delta_0} & 0 & 0 \\ 0 & 0 & I_{n_1} \end{bmatrix},$$

where $T \in \mathbb{C}^{\delta_0\times\delta_0}$ is nonsingular and $n_1 = m-n_0-2\delta_0 = m-n-\delta_0$. With $X_3 = X_1(I_{n_0}\oplus X_2)$ we then have

$$X_3^TBY = \begin{bmatrix} I_{n_0} & 0 \\ 0 & T \\ 0 & 0_{\delta_0} \\ 0 & 0 \end{bmatrix}, \quad X_3^TX_3 = I_{n_0} \oplus \begin{bmatrix} 0 & I_{\delta_0} \\ I_{\delta_0} & 0 \end{bmatrix} \oplus I_{n_1}.$$

Let P be the permutation that rearranges the block rows of X_3^TBY in the order $4, 3, 1, 2$ and let $X = X_3P^T$. Then

$$X^TBY = \begin{bmatrix} 0 & 0 \\ 0 & 0_{\delta_0} \\ I_{n_0} & 0 \\ 0 & T \end{bmatrix}, \quad X^TX = I_{n_1} \oplus \begin{bmatrix} 0 & 0 & I_{\delta_0} \\ 0 & I_{n_0} & 0 \\ I_{\delta_0} & 0 & 0 \end{bmatrix}.$$

Post-multiplying Y^{-1} to the first of these two equations and setting

$$B_0 = \begin{bmatrix} I_{n_0} & 0 \\ 0 & T \end{bmatrix} Y^{-1},$$

we have the asserted form. □

In the previous results we have obtained factorizations for matrices B such that B^TB is the identity or zero. We get similar results if $B^TJ_mB = J_n$ or $B^TJ_mB = 0$.

Lemma 3.4 *If $B \in \mathbb{C}^{2m\times 2n}$ satisfies $B^T J_m B = J_n$, then $m \geq n$ and there exists a nonsingular matrix $X \in \mathbb{C}^{2m\times 2m}$ such that*

$$X^T B_1 = \begin{bmatrix} I_n & 0 \\ 0 & 0 \\ 0 & I_n \\ 0 & 0 \end{bmatrix}, \quad X^T J_m X = J_m.$$

Proof. The proof is similar to that for Lemma 3.1 and is hence omitted. □

Lemma 3.5 *Let $b \in \mathbb{C}^{2m}$. Then there is a unitary matrix $X \in \mathbb{C}^{2m\times 2m}$ such that*

$$X^T b = \alpha e_1, \qquad X^T J_m X = J_m.$$

Proof. We again present a constructive proof that can be implemented into a numerical algorithm. Let $b = [b_1^T, b_2^T]^T$ with $b_1, b_2 \in \mathbb{C}^m$ and let $H_2 \in \mathbb{C}^{m\times m}$ be a unitary matrix (e.g. a Householder matrix) such that

$$H_2^T b_2 = \beta e_1.$$

With $H_2^{-1} b_1 = [b_{11}, \ldots, b_{m1}]^T$ one then can determine (e.g. via a QR factorization) a unitary matrix

$$G = \frac{1}{\widetilde{b}_{11}} \begin{bmatrix} \bar{b}_{11} & -\beta \\ \bar{\beta} & b_{11} \end{bmatrix}, \quad \widetilde{b}_{11} = \sqrt{|b_{11}|^2 + |\beta|^2}, \quad \text{such that } G^T \begin{bmatrix} b_{11} \\ \beta \end{bmatrix} = \begin{bmatrix} \widetilde{b}_{11} \\ 0 \end{bmatrix}.$$

Note that $G^T J_2 G = J_2$. Next, determine a unitary matrix $H_1 \in \mathbb{C}^{m\times m}$ such that

$$H_1^T [\widetilde{b}_{11}, b_{21}, \ldots, b_{m1}]^T = \alpha e_1.$$

Finally, let

$$X = \begin{bmatrix} H_2^{-T} & 0 \\ 0 & H_2 \end{bmatrix} \hat{G} \begin{bmatrix} H_1 & 0 \\ 0 & H_1^{-T} \end{bmatrix},$$

where $\hat{G} \in \mathbb{C}^{2m\times 2m}$ is the unitary matrix obtained by replacing the $(1,1)$, $(1, m+1)$, $(m+1, 1)$, and $(m+1, m+1)$ elements of the identity matrix I_{2m} with the corresponding elements of G, respectively. It is easily verified that X is unitary and satisfies $X^T b = \alpha e_1$ and $X^T J_m X = J_m$. □

Lemma 3.6 *If $B \in \mathbb{C}^{2m\times n}$ satisfies* $\operatorname{rank} B = n$ *and $B^T J_m B = 0$, then $m \geq n$ and there exists a unitary matrix $X \in \mathbb{C}^{2m\times 2m}$ such that*

$$X^T B = \begin{bmatrix} B_0 \\ 0 \end{bmatrix}, \quad X^T J_m X = J_m,$$

where $B_0 \in \mathbb{C}^{n\times n}$ is upper triangular invertible.

Proof. By Lemma 3.5, there exists a unitary matrix X_1 such that

$$X_1^T B = \begin{bmatrix} b_{11} & b_1^T \\ 0 & B_{22} \\ 0 & b_3^T \\ 0 & B_{24} \end{bmatrix}, \quad X_1^T J_m X_1 = J_m,$$

where $b_1, b_3 \in \mathbb{C}^{n-1}$. Since rank $B = n$, we have $b_{11} \neq 0$ and from

$$(X_1 B)^T J_m (X_1 B) = B^T J_m B = 0,$$

it follows that

$$b_3 = 0, \qquad \begin{bmatrix} B_{22} \\ B_{24} \end{bmatrix}^T J_{m-1} \begin{bmatrix} B_{22} \\ B_{24} \end{bmatrix} = 0.$$

Applying the same procedure inductively to $\begin{bmatrix} B_{22} \\ B_{24} \end{bmatrix}$, we obtain a unitary matrix X such that

$$X^T B = \begin{bmatrix} B_0 \\ 0 \end{bmatrix} \begin{matrix} n \\ 2m-n \end{matrix}, \qquad X^T J_m X = J_m,$$

where $B_0 \in \mathbb{C}^{n \times n}$ is upper triangular and invertible. □

Proposition 3.7 *Let $B \in \mathbb{C}^{2m \times n}$. Suppose that* rank $B = n$, rank $B^T J_m B = 2n_0 \leq n$, *i.e., $\delta_0 = n - 2n_0$ is the dimension of the null space of $B^T J_m B$. Then there exists an invertible matrix $X \in \mathbb{C}^{2m \times 2m}$ such that*

$$X^T B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix} \begin{matrix} 2m-n \\ n \end{matrix}, \quad X^T J_m X = J_{n_1} \oplus \begin{bmatrix} 0 & 0 & I_{\delta_0} \\ 0 & J_{n_0} & 0 \\ -I_{\delta_0} & 0 & 0 \end{bmatrix},$$

where $B_0 \in \mathbb{C}^{n \times n}$ is nonsingular and $n_1 = m - n_0 - \delta_0$.

Proof. Since $B^T J_m B$ is complex skew-symmetric, by the assumption and Corollary 2.5 there exists a nonsingular matrix $Y \in \mathbb{C}^{n \times n}$ such that

$$Y^T B^T J_m B Y = \begin{bmatrix} J_{n_0} & 0 \\ 0 & 0_{\delta_0} \end{bmatrix}.$$

Let $B_1 \in \mathbb{C}^{2m \times 2n_0}$ be the matrix formed by the leading $2n_0$ columns of BY. By Lemma 3.4 there exists a nonsingular $X_1 \in \mathbb{C}^{2m \times 2m}$ such that

$$X_1^T B_1 = \begin{bmatrix} I_{n_0} & 0 \\ 0 & 0 \\ 0 & I_{n_0} \\ 0 & 0 \end{bmatrix}, \quad X_1^T J_m X_1 = J_m.$$

We have

$$X_1^T B Y = \begin{bmatrix} I_{n_0} & 0 & B_{13} \\ 0 & 0 & B_{23} \\ 0 & I_{n_0} & B_{33} \\ 0 & 0 & B_{43} \end{bmatrix}.$$

Since $X_1^T J_m X_1 = J_m$ also implies $X_1 J_m X_1^T = J_m$, from

$$(X_1^T B Y)^T J_m (X_1^T B Y) = Y^T B^T J_m B Y = \begin{bmatrix} J_{n_0} & 0 \\ 0 & 0_{\delta_0} \end{bmatrix},$$

we obtain that

$$B_{13} = 0, \quad B_{33} = 0, \quad \begin{bmatrix} B_{23} \\ B_{43} \end{bmatrix}^T J_{m-n_0} \begin{bmatrix} B_{23} \\ B_{43} \end{bmatrix} = 0_{\delta_0}.$$

Since B has full column rank, so does $\begin{bmatrix} B_{23} \\ B_{43} \end{bmatrix}$. By Lemma 3.6, there exists an invertible $X_2 \in \mathbb{C}^{(2m-2n_0)\times(2m-2n_0)}$ such that

$$X_2^T \begin{bmatrix} B_{23} \\ B_{43} \end{bmatrix} = \begin{bmatrix} \widetilde{B}_0 \\ 0 \end{bmatrix}, \quad X_2^T J_{m-n_0} X_2 = J_{m-n_0},$$

where $\widetilde{B}_0 \in \mathbb{C}^{\delta_0 \times \delta_0}$ is invertible. Let P_1 be a permutation that interchanges the second and third block rows of $X_1^T BY$ and set $X_3 = X_1 P_1^T (I_{2n_0} \oplus X_2)$. Then

$$X_3^T BY = \begin{bmatrix} I_{2n_0} & 0 \\ 0 & \widetilde{B}_0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} 2n_0 \\ \delta_0 \\ n_1 \\ \delta_0 \\ n_1 \end{matrix}, \qquad X_3^T J_m X_3 = J_{n_0} \oplus J_{m-n_0},$$

where $n_1 = m - n_0 - \delta_0$. (For convenience, we have split the zero block row of $X_3^T BY$ into three block rows.) Let P be a permutation that changes the block rows of $X_3^T BY$ to the order $3, 5, 4, 1, 2$ by pre-multiplication, and let $X = X_3 P^T (I_{2n_1} \oplus (-I_{\delta_0}) \oplus I_{2n_0+\delta_0})$. Then

$$X^T BY = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I_{2n_0} & 0 \\ 0 & \widetilde{B}_0 \end{bmatrix} \begin{matrix} 2n_1 \\ \delta_0 \\ 2n_0 \\ \delta_0 \end{matrix}, \quad X^T J_m X = J_{n_1} \oplus \begin{bmatrix} 0 & 0 & I_{\delta_0} \\ 0 & J_{n_0} & 0 \\ -I_{\delta_0} & 0 & 0 \end{bmatrix}.$$

Post-multiplying Y^{-1} to the first equation and setting $B_0 = (I_{2n_0} \oplus \widetilde{B}_0)Y^{-1}$, we have the asserted form. □

In this section we have presented preliminary factorizations that will form the basis in determining the canonical forms in the following sections.

4 Canonical form for $G, \hat{G}$ complex symmetric

We start with the case that the matrix A under consideration is square and nonsingular. If $\Sigma = U^* AV$ is the standard singular decomposition of A, then $U^* AA^* U = V^* A^* AV = \Sigma^2$, i.e., the canonical forms for both AA^* and A^*A are just the square of the canonical form for A. This fact has a generalization in the case of a matrix triple $(A, G, \hat{G})$, where G and $\hat{G}$ are complex symmetric. To start from a square root of the $\hat{G}$-symmetric matrix $\hat{\mathcal{H}} = \hat{G}^{-1} A^T G^{-1} A$ will be the key strategy in the derivation of the canonical form in the following result.

Theorem 4.1 *Let $A \in \mathbb{C}^{n\times n}$ be nonsingular and let $G, \hat{G} \in \mathbb{C}^{n\times n}$ be complex symmetric and nonsingular. Then there exist nonsingular matrices $X, Y \in \mathbb{C}^{n\times n}$ such that*

$$\begin{aligned} X^T AY &= \mathcal{J}_{\xi_1}(\mu_1) \oplus \cdots \oplus \mathcal{J}_{\xi_m}(\mu_m), \\ X^T GX &= R_{\xi_1} \oplus \cdots \oplus R_{\xi_m}, \\ Y^T \hat{G} Y &= R_{\xi_1} \oplus \cdots \oplus R_{\xi_m}, \end{aligned} \tag{4.1}$$

where $\mu_j \in \mathbb{C} \setminus \{0\}$*,* $\arg \mu_j \in [0,\pi)$*, and* $\xi_j \in \mathbb{N}$ *for* $j = 1,\ldots,m$*. Moreover, for the* $\hat{G}$*-symmetric matrix* $\hat{\mathcal{H}} = \hat{G}^{-1}A^TG^{-1}A$ *and for the G-symmetric matrix* $\mathcal{H} = G^{-1}A\hat{G}^{-1}A^T$ *we have that*

$$\begin{aligned} Y^{-1}\hat{\mathcal{H}}Y &= \mathcal{J}_{\xi_1}^2(\mu_1) \oplus \cdots \oplus \mathcal{J}_{\xi_m}^2(\mu_m), \\ X^{-1}\mathcal{H}X &= \mathcal{J}_{\xi_1}^2(\mu_1)^T \oplus \cdots \oplus \mathcal{J}_{\xi_m}^2(\mu_m)^T. \end{aligned} \tag{4.2}$$

Moreover, the form (4.1) *is unique up to the simultaneous permutation of blocks in the right hand side of* (4.1)*.*

Proof. By Lemma 2.10, $\hat{\mathcal{H}}$ has a unique $\hat{G}$-symmetric square root $S \in \mathbb{C}^{n\times n}$ satisfying $\sigma(S) \subseteq \{\mu \in \mathbb{C} \setminus \{0\} : \arg(\mu) \in [0,\pi)\}$. Then by Theorem 2.6, there exists a nonsingular matrix $\widetilde{Y} \in \mathbb{C}^{n\times n}$ such that

$$\begin{aligned} S_{\text{CF}} := \widetilde{Y}^{-1}S\widetilde{Y} &= \mathcal{J}_{\xi_1}(\mu_1) \oplus \cdots \oplus \mathcal{J}_{\xi_m}(\mu_m), \\ G_{\text{CF}} := \widetilde{Y}^T\hat{G}\widetilde{Y} &= R_{\xi_1} \oplus \cdots \oplus R_{\xi_m}, \\ \mathcal{H}_{\text{CF}} := \widetilde{Y}^{-1}\hat{\mathcal{H}}\widetilde{Y} &= \mathcal{J}_{\xi_1}^2(\mu_1) \oplus \cdots \oplus \mathcal{J}_{\xi_m}^2(\mu_m), \end{aligned}$$

where $\mu_j \in \mathbb{C} \setminus \{0\}$, $\arg \mu_j \in [0,\pi)$, and $\xi_j \in \mathbb{N}$ for $j = 1,\ldots,m$. (Here, the third line immediately follows from $\hat{\mathcal{H}} = S^2$). Using $G^{-1}A\hat{\mathcal{H}} = \mathcal{H}G^{-1}A$ and the fact that $G^{-1}A$ is nonsingular, we find that $\hat{\mathcal{H}}$ and $\mathcal{H}$ are similar. Since the canonical form of G-symmetric matrices in Theorem 2.6 is uniquely determined by the Jordan canonical form, we obtain from Theorem 2.6 that the canonical forms of the pairs $(\hat{\mathcal{H}},\hat{G})$ and $(\mathcal{H},G)$ coincide. In particular, this implies the existence of a nonsingular matrix $\widetilde{X} \in \mathbb{C}^{n\times n}$ such that

$$\begin{aligned} \mathcal{H}_{\text{CF}} = \widetilde{X}^{-1}\mathcal{H}\widetilde{X} &= \mathcal{J}_{\xi_1}^2(\mu_1) \oplus \cdots \oplus \mathcal{J}_{\xi_m}^2(\mu_m), \\ G_{\text{CF}} = \widetilde{X}^TG\widetilde{X} &= R_{\xi_1} \oplus \cdots \oplus R_{\xi_m}. \end{aligned}$$

Finally setting $X = G^{-1}\widetilde{X}^{-T}$ and $Y = A^{-1}G\widetilde{X}S_{\text{CF}}$, we obtain

$$\begin{aligned} X^TAY &= \widetilde{X}^{-1}G^{-1}AA^{-1}G\widetilde{X}S_{\text{CF}} = S_{\text{CF}} \\ X^TGX &= \widetilde{X}^{-1}G^{-1}GG^{-1}\widetilde{X}^{-T} = (\widetilde{X}^TG\widetilde{X})^{-1} = G_{1,\text{CF}}^{-1} = G_{\text{CF}} \\ Y^T\hat{G}Y &= S_{\text{CF}}^T\widetilde{X}^TGA^{-T}\hat{G}A^{-1}G\widetilde{X}S_{\text{CF}} \\ &= S_{\text{CF}}^T\widetilde{X}^TG\widetilde{X}\widetilde{X}^{-1}\mathcal{H}^{-1}\widetilde{X}S_{\text{CF}} \\ &= S_{\text{CF}}^TG_{\text{CF}}(\mathcal{H}_{\text{CF}})^{-1}S_{\text{CF}} = G_{\text{CF}}S_{\text{CF}}(\mathcal{H}_{\text{CF}})^{-1}S_{\text{CF}} = G_{\text{CF}} \end{aligned}$$

as desired, where we used that S_{CF} is G_{CF}-symmetric and that $S_{\text{CF}}^2 = \mathcal{H}_{\text{CF}}$. It is now easy to check that $Y^{-1}\hat{\mathcal{H}}Y$ and $X^{-1}\mathcal{H}X$ have the claimed forms. Concerning uniqueness, we note that the form (4.1) is already uniquely determined by the Jordan structure of $\hat{\mathcal{H}}$ and by the restriction $\mu_j \in \mathbb{C} \setminus \{0\}$, $\arg \mu_j \in [0,\pi)$. □

The canonical form for the case that A is singular or rectangular is more involved, because then the matrices $\hat{\mathcal{H}}$ and $\mathcal{H}$ may be singular as well. The key idea in the proof of Theorem 4.1 was the construction of a $\hat{G}$-symmetric square root of $\hat{\mathcal{H}}$, but if $\hat{\mathcal{H}}$ is singular, then such a square root need not exist. (For example, the R_n-symmetric nilpotent matrix $\mathcal{J}_n(0)$ does not have any square root let alone a R_n-symmetric one.) A second difficulty comes from the fact that the Jordan structures of $\hat{\mathcal{H}}$ and $\mathcal{H}$ may be different. For example, if

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad G = R_2 \oplus R_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \hat{G} = R_1 \oplus R_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

then we obtain that

$$\hat{\mathcal{H}} = \hat{G}^{-1}A^TG^{-1}A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{H} = G^{-1}A\hat{G}^{-1}A^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Here $\hat{\mathcal{H}}$ has a 1×1 and a 3×3 Jordan block associated with the eigenvalue zero, while $\mathcal{H}$ has two 2×2 Jordan blocks associated with zero. In general, we obtain the following result.

Theorem 4.2 *Let $A \in \mathbb{C}^{m\times n}$ and let $G \in \mathbb{C}^{m\times m}, \hat{G} \in \mathbb{C}^{n\times n}$ be complex symmetric and nonsingular. Then there exist nonsingular matrices $X \in \mathbb{C}^{m\times m}$ and $Y \in \mathbb{C}^{n\times n}$ such that*

$$\begin{aligned} X^TAY &= A_c \oplus A_{z,1} \oplus A_{z,2} \oplus A_{z,3} \oplus A_{z,4}, \\ X^TGX &= G_c \oplus G_{z,1} \oplus G_{z,2} \oplus G_{z,3} \oplus G_{z,4}, \\ Y^T\hat{G}Y &= \hat{G}_c \oplus \hat{G}_{z,1} \oplus \hat{G}_{z,2} \oplus \hat{G}_{z,3} \oplus \hat{G}_{z,4}. \end{aligned} \tag{4.3}$$

Moreover, for the $\hat{G}$-symmetric matrix $\hat{\mathcal{H}} = \hat{G}^{-1}A^TG^{-1}A \in \mathbb{C}^{n\times n}$ and for the G-symmetric matrix $\mathcal{H} = G^{-1}A\hat{G}^{-1}A^T \in \mathbb{C}^{m\times m}$ we have that

$$\begin{aligned} Y^{-1}\hat{\mathcal{H}}Y &= \hat{\mathcal{H}}_c \oplus \hat{\mathcal{H}}_{z,1} \oplus \hat{\mathcal{H}}_{z,2} \oplus \hat{\mathcal{H}}_{z,3} \oplus \hat{\mathcal{H}}_{z,4}, \\ X^{-1}\mathcal{H}X &= \mathcal{H}_c \oplus \mathcal{H}_{z,1} \oplus \mathcal{H}_{z,2} \oplus \mathcal{H}_{z,3} \oplus \mathcal{H}_{z,4}. \end{aligned}$$

The diagonal blocks in these decompositions have the following forms:

0) *blocks associated with nonzero eigenvalues of $\hat{\mathcal{H}}$ and $\mathcal{H}$:*
   $A_c, G_c, \hat{G}_c$ have the forms as in (4.1) and $\hat{\mathcal{H}}_c, \mathcal{H}_c$ have the forms as in (4.2);

1) *one block corresponding to n_0 Jordan blocks of size 1×1 of $\hat{\mathcal{H}}$ and m_0 Jordan blocks of size 1×1 of $\mathcal{H}$ associated with the eigenvalue zero:*

$$A_{z,1} = 0_{m_0\times n_0}, \quad G_{z,1} = I_{m_0}, \quad \hat{G}_{z,1} = I_{n_0}, \quad \hat{\mathcal{H}}_{z,1} = 0_{n_0}, \quad \mathcal{H}_{z,1} = 0_{m_0},$$

   where $m_0, n_0 \in \mathbb{N} \cup \{0\}$;

2) *blocks corresponding to a pair of $j \times j$ Jordan blocks of $\hat{\mathcal{H}}$ and $\mathcal{H}$ associated with the eigenvalue zero:*

$$\begin{aligned} A_{z,2} &= \bigoplus_{i=1}^{\gamma_1} \mathcal{J}_2(0) \oplus \bigoplus_{i=1}^{\gamma_2} \mathcal{J}_4(0) \oplus \cdots \oplus \bigoplus_{i=1}^{\gamma_\ell} \mathcal{J}_{2\ell}(0)\ , \\ G_{z,2} &= \bigoplus_{i=1}^{\gamma_1} R_2 \oplus \bigoplus_{i=1}^{\gamma_2} R_4 \oplus \cdots \oplus \bigoplus_{i=1}^{\gamma_\ell} R_{2\ell}\ , \\ \hat{G}_{z,2} &= \bigoplus_{i=1}^{\gamma_1} R_2 \oplus \bigoplus_{i=1}^{\gamma_2} R_4 \oplus \cdots \oplus \bigoplus_{i=1}^{\gamma_\ell} R_{2\ell}\ , \\ \hat{\mathcal{H}}_{z,2} &= \bigoplus_{i=1}^{\gamma_1} \mathcal{J}_2^2(0) \oplus \bigoplus_{i=1}^{\gamma_2} \mathcal{J}_4^2(0) \oplus \cdots \oplus \bigoplus_{i=1}^{\gamma_\ell} \mathcal{J}_{2\ell}^2(0)\ , \\ \mathcal{H}_{z,2} &= \bigoplus_{i=1}^{\gamma_1} \mathcal{J}_2^2(0)^T \oplus \bigoplus_{i=1}^{\gamma_2} \mathcal{J}_4^2(0)^T \oplus \cdots \oplus \bigoplus_{i=1}^{\gamma_\ell} \mathcal{J}_{2\ell}^2(0)^T, \end{aligned}$$

   where $\gamma_1, \ldots, \gamma_\ell \in \mathbb{N} \cup \{0\}$; thus, $\hat{\mathcal{H}}_{z,2}$ and $\mathcal{H}_{z,2}$ both have each $2\gamma_j$ Jordan blocks of size $j \times j$ for $j = 1, \ldots, \ell$;

3) *blocks corresponding to a $j \times j$ Jordan block of $\hat{\mathcal{H}}$ and a $(j+1)\times(j+1)$ Jordan block of $\mathcal{H}$ associated with the eigenvalue zero:*

$$\begin{aligned}
A_{z,3} &= \bigoplus_{i=1}^{m_1} \begin{bmatrix} I_1 \\ 0 \end{bmatrix}_{2\times 1} \oplus \bigoplus_{i=1}^{m_2} \begin{bmatrix} I_2 \\ 0 \end{bmatrix}_{3\times 2} \oplus \cdots \oplus \bigoplus_{i=1}^{m_{\ell-1}} \begin{bmatrix} I_{\ell-1} \\ 0 \end{bmatrix}_{\ell\times(\ell-1)}, \\
G_{z,3} &= \bigoplus_{i=1}^{m_1} R_2 \oplus \bigoplus_{i=1}^{m_2} R_3 \oplus \cdots \oplus \bigoplus_{i=1}^{m_{\ell-1}} R_\ell , \\
\hat{G}_{z,3} &= \bigoplus_{i=1}^{m_1} R_1 \oplus \bigoplus_{i=1}^{m_2} R_2 \oplus \cdots \oplus \bigoplus_{i=1}^{m_{\ell-1}} R_{\ell-1} , \\
\hat{\mathcal{H}}_{z,3} &= \bigoplus_{i=1}^{m_1} \mathcal{J}_1(0) \oplus \bigoplus_{i=1}^{m_2} \mathcal{J}_2(0) \oplus \cdots \oplus \bigoplus_{i=1}^{m_{\ell-1}} \mathcal{J}_{\ell-1}(0) , \\
\mathcal{H}_{z,3} &= \bigoplus_{i=1}^{m_1} \mathcal{J}_2(0)^T \oplus \bigoplus_{i=1}^{m_2} \mathcal{J}_3(0)^T \oplus \cdots \oplus \bigoplus_{i=1}^{m_{\ell-1}} \mathcal{J}_\ell(0)^T,
\end{aligned}$$

where $m_1,\ldots,m_{\ell-1} \in \mathbb{N}\cup\{0\}$; thus, $\hat{\mathcal{H}}_{z,3}$ has m_j Jordan blocks of size $j\times j$ and $\mathcal{H}_{z,3}$ has m_j Jordan blocks of size $(j+1)\times(j+1)$ for $j=1,\ldots,\ell-1$;

4) *blocks corresponding to a $(j+1)\times(j+1)$ Jordan block of $\hat{\mathcal{H}}$ and a $j\times j$ Jordan block of $\mathcal{H}$ associated with the eigenvalue zero:*

$$\begin{aligned}
A_{z,4} &= \bigoplus_{i=1}^{n_1} \begin{bmatrix} 0 & I_1 \end{bmatrix}_{1\times 2} \oplus \bigoplus_{i=1}^{n_2} \begin{bmatrix} 0 & I_2 \end{bmatrix}_{2\times 3} \oplus \cdots \oplus \bigoplus_{i=1}^{n_{\ell-1}} \begin{bmatrix} 0 & I_{\ell-1} \end{bmatrix}_{(\ell-1)\times\ell}, \\
G_{z,4} &= \bigoplus_{i=1}^{n_1} R_1 \oplus \bigoplus_{i=1}^{n_2} R_2 \oplus \cdots \oplus \bigoplus_{i=1}^{n_{\ell-1}} R_{\ell-1} , \\
\hat{G}_{z,4} &= \bigoplus_{i=1}^{n_1} R_2 \oplus \bigoplus_{i=1}^{n_2} R_3 \oplus \cdots \oplus \bigoplus_{i=1}^{n_{\ell-1}} R_\ell , \\
\hat{\mathcal{H}}_{z,4} &= \bigoplus_{i=1}^{n_1} \mathcal{J}_2(0) \oplus \bigoplus_{i=1}^{n_2} \mathcal{J}_3(0) \oplus \cdots \oplus \bigoplus_{i=1}^{n_{\ell-1}} \mathcal{J}_\ell(0) , \\
\mathcal{H}_{z,4} &= \bigoplus_{i=1}^{n_1} \mathcal{J}_1(0)^T \oplus \bigoplus_{i=1}^{n_2} \mathcal{J}_2(0)^T \oplus \cdots \oplus \bigoplus_{i=1}^{n_{\ell-1}} \mathcal{J}_{\ell-1}(0)^T,
\end{aligned}$$

where $n_1,\ldots,n_{\ell-1} \in \mathbb{N}\cup\{0\}$; thus, $\hat{\mathcal{H}}_{z,4}$ has n_j Jordan blocks of size $(j+1)\times(j+1)$ and $\mathcal{H}_{z,4}$ has n_j Jordan blocks of size $j\times j$ for $j=1,\ldots,\ell-1$;

For the eigenvalue zero, the matrices $\hat{\mathcal{H}}$ and $\mathcal{H}$ have $2\gamma_j+m_j+n_{j-1}$ respectively $2\gamma_j+m_{j-1}+n_j$ Jordan blocks of size $j\times j$ for $j=1,\ldots,\ell$, where $m_\ell=n_\ell=0$ and where ℓ is the maximum of the indices of $\hat{\mathcal{H}}$ and $\mathcal{H}$. (Here, index *refers to the size of the largest Jordan block associated with the eigenvalue zero.)*

Moreover, the form (4.3) *is unique up to simultaneous block permutation of the blocks in the diagonal blocks of the right hand side of* (4.3)*.*

Proof. The proof is very long and technical and is therefore postponed to the Appendix. □

We highlight that the numbers m_0 and n_0 in 1) of Theorem 4.2 are allowed to be zero. This has the effect that there may occur rectangular matrices with a total number of zero rows or columns in the canonical form. We illustrate this phenomenon with the following example.

Example 4.3 Consider the two non-equivalent triples

$$A_1 = \left[\begin{array}{cc} 0 & 1 \end{array}\right],\ G_1 = \left[\begin{array}{c} 1 \end{array}\right],\ \hat{G}_1 = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right]$$

and $$A_2 = \left[\begin{array}{cc} 0 & 1 \end{array}\right],\ G_2 = \left[\begin{array}{c} 1 \end{array}\right],\ \hat{G}_2 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right].$$

The first example is just one block of type 4) in Theorem 4.2. Indeed, forming the products

$$\hat{\mathcal{H}}_1 = \hat{G}_1^{-1} A_1^T G_1^{-1} A = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right], \quad \mathcal{H}_1 = G_1^{-1} A \hat{G}_1^{-1} A_1^T = \left[\begin{array}{c} 0 \end{array}\right],$$

we see that, as predicted by Theorem 4.2, $\hat{\mathcal{H}}_1$ has only one Jordan block of size 2 associated with the eigenvalue $\lambda = 0$ whereas $\mathcal{H}_1$ has one Jordan block of size 1 associated with $\lambda = 0$. The situation is different in the second case. Here, we obtain

$$\hat{\mathcal{H}}_2 = \hat{G}_2^{-1} A_2^T G_2^{-1} A = \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right], \quad \mathcal{H}_2 = G_2^{-1} A \hat{G}_2^{-1} A_2^T = \left[\begin{array}{c} 1 \end{array}\right],$$

i.e., $\hat{\mathcal{H}}_2$ has two Jordan blocks of size 1, one associated with $\lambda = 0$ and a second one associated with $\lambda = 1$, while $\mathcal{H}_2$ has one Jordan block of size 1 associated with $\lambda = 1$. Here, the triple $(A_2, G_2, \hat{G}_2)$ is in canonical form consisting of one block of type 1) and size 0×1 and of one block of type 0):

$$A_2 = \left[\begin{array}{c|c} 0 & 1 \end{array}\right],\ G_2 = \left[\begin{array}{c|c} & 1 \end{array}\right],\ \hat{G}_2 = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right].$$

Remark 4.4 Theorem 4.2 in particular covers the special case $G = I_m$ and $\hat{G} = I_n$, i.e., the case that $\hat{\mathcal{H}} = A^T A$ and $\mathcal{H} = AA^T$. In comparison to the *standard singular values* of a matrix $A \in \mathbb{C}^{m \times n}$ which are $\sigma_1, \ldots, \sigma_{\min(m,n)} \geq 0$ and which are the square roots of the eigenvalues of AA^* and A^*A, we now obtain the "*transpose singular values*" of A according to Theorem 4.2 as

$$\mathcal{J}_{\xi_1}(\mu_1), ..., 0_{m_0 \times n_0}, \mathcal{J}_{2p_1}(0), ..., \left[\begin{array}{c} I_{q_1} \\ 0 \end{array}\right], ..., \left[\begin{array}{cc} 0 & I_{r_1} \end{array}\right], ...,$$

where $\mu_j \neq 0$, $\arg(\mu_j) \in [0, \pi)$ and $\xi_j, p_j, q_j, r_j \in \mathbb{N}$. Theorem 4.2 displays how thee blocks are related to the eigenvalues and Jordan structures of AA^T and $A^T A$.

The canonical form of A in Theorem 4.2 together with the canonical forms for AA^T and $A^T A$ in the special case $G = I_m$, $\hat{G} = I_n$ can also be deduced from Theorem 5 in [11], where the canonical form for a pair (B, C), $B \in \mathbb{C}^{m \times n}$, $C \in \mathbb{C}^{n \times m}$ under the transformation

$$(B, C) \mapsto (X^{-1} B Y, Y^{-1} C X), \quad X, Y \text{ nonsingular}$$

is given. Setting then $B = A$ and $C = A^T$ then yields the desired form. The result of Theorem 4.2, however, gives additional information on the transformation matrices X and Y, because we also have a canonical form for $X^T X = X^T G X$ and $Y^T Y = Y^T \hat{G} Y$.

A well known result by Flanders [4] completely describes the Jordan structures of the products BC and CB, where $B \in \mathbb{C}^{m \times n}$ and $C \in \mathbb{C}^{n \times m}$. Recall that the *partial multiplicities* of an eigenvalue λ of a matrix $M \in \mathbb{C}^{n \times n}$ are just the sizes of the Jordan blocks associated with λ in the Jordan canonical form for M.

Theorem 4.5 ([4]) *For $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$ the following conditions are equivalent:*

1) *There exist matrices $B \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{n\times m}$ such that $M = BC$ and $N = CB$.*

2) *M and N satisfy the* Flanders condition, *i.e.,*

   i) *M and N have the same nonzero eigenvalues and their algebraic, geometric, and partial multiplicities coincide.*

   ii) *If $(\tau_i)_{i\in\mathbb{N}}$ is the monotonically decreasing sequence of partial multiplicities of M associated with the eigenvalue zero, made infinite by adjunction of zeros, and if $(\zeta_i)_{i\in\mathbb{N}}$ is the corresponding sequence of N, then $|\tau_i - \zeta_i| \leq 1$ for all $i \in \mathbb{N}$.*

With the canonical form of Theorem 4.2, we are now able to prove a specialization of Theorem 4.5 for the case of complex symmetric matrices.

Theorem 4.6 *For $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$ the following conditions are equivalent:*

1) *There exists a matrix $A \in \mathbb{C}^{m\times n}$ such that $M = AA^T$ and $N = A^TA$.*

2) *M and N are symmetric and satisfy the Flanders condition, i.e.,* i) *and* ii) *in Theorem* 4.5*, as well as*

   iii) *Let ϕ_k be the number of indices j for which $\tau_j = \zeta_j = k$, where $(\tau_i)_{i\in\mathbb{N}}$ and $(\zeta_i)_{i\in\mathbb{N}}$ are the sequences as in Theorem* 4.5*, and let $k_1 > \cdots > k_\nu$ be the numbers $k \in \mathbb{N}$ for which ϕ_k is odd. If ν is even, then for $j = 1, \ldots, \lceil \frac{\nu}{2} \rceil$ we have that $\phi_k \neq 0$ for all k with $k_{2j-1} \geq k \geq k_{2j}$. (Here, $\lceil \kappa \rceil$ denotes the smallest integer larger or equal to κ and we set $k_{\nu+1} := 1$ in the case that ν is odd.)*

Proof. '1) $\Rightarrow$ 2)': Let $\mathcal{H} = M = AA^T$ and $\hat{\mathcal{H}} = N = A^TA$ and let ω_j and $\hat{\omega}_j$ denote the number of Jordan blocks of size $j \times j$ associated with the eigenvalue zero of $\mathcal{H}$ and $\hat{\mathcal{H}}$, respectively. Using the same notation as in Theorem 4.2, we obtain that

$$\omega_j = 2\gamma_j + n_j + m_{j-1} \quad \text{and} \quad \hat{\omega}_j = 2\gamma_j + m_j + n_{j-1}, \quad j = 1, \ldots, \ell.$$

Assume without loss of generality that $m_{\ell-1} \geq n_{\ell-1}$. Since $m_\ell = n_\ell = 0$, we find that the first $2\gamma_\ell + n_{\ell-1}$ entries in the sequences $(\tau_i)_{i\in\mathbb{N}}$ and $(\zeta_i)_{i\in\mathbb{N}}$ are given by ℓ which implies $\phi_\ell = 2\gamma_\ell + n_{\ell-1}$. The sequence (τ_i) has $m_{\ell-1} - n_{\ell-1}$ more entries equal to ℓ that are paired to $m_{\ell-1} - n_{\ell-1}$ entries $\ell - 1$ in (ζ_i). Since then there are $2\gamma_{\ell-1} + n_{\ell-1} + n_{\ell-2}$ more entries $\ell - 1$ in (ζ_i) and $2\gamma_{\ell-1} + n_{\ell-1} + m_{\ell-2}$ entries $\ell - 1$ in (τ_i), we obtain that $\phi_{\ell-1} = 2\gamma_{\ell-1} + n_{\ell-1} + \min(m_{\ell-2}, n_{\ell-2})$. Continuing the counting in the way just described finally yields

$$\phi_j = 2\gamma_j + \min(m_j, n_j) + \min(m_{j-1}, n_{j-1}), \quad j = 1, \ldots, \ell. \tag{4.4}$$

If $\nu = 0$ then there is nothing to prove, so assume $\nu \geq 1$. Since $0 = \min(m_\ell, n_\ell)$ is even as well as $\phi_\ell, \ldots, \phi_{k_1+1}$, we obtain from (4.4) that $\min(m_{j-1}, n_{j-1})$ is even for $j > k_1$ and $\min(m_{k_1-1}, n_{k_1-1})$ is odd. Clearly, we must then have that $\min(m_{k-1}, n_{k-1})$ is odd for all k satisfying $k_1 > k > k_2$. In particular, this implies $\phi_k \neq 0$ for all such k as well as $\phi_{k_1} \neq 0$ and $\phi_{k_2} \neq 0$. If $\nu \leq 2$ we are done. Otherwise, $\min(m_{k_2-1}, n_{k_2-1})$ is even and we can repeat the argument for $k_{2j-1} \geq k \geq k_{2j}$ for $j = 2, \ldots, \lceil \frac{\nu}{2} \rceil$.

‘2) ⇒ 1)’: Let ℓ be the largest entry that appears in one of the sequences $(\tau_i)_{i\in\mathbb{N}}$ and $(\zeta_i)_{i\in\mathbb{N}}$. First, let us assume that $\nu = 0$ or $k_1 = 1$, i.e., ϕ_j is even for $j = 2, \dots, \ell$. Then we build up a matrix triple $(\widetilde{A}, G, \hat{G})$ as a direct sum of blocks as follows: for the ϕ_k indices j with $\tau_j = \zeta_j = k$, $k \neq 1$, we take $\phi_k/2$ blocks as in 2) of Theorem 4.2 and for each index j with $|\tau_j - \zeta_j| = 1$, $\tau_j, \zeta_j \neq 0$, we take a block as in 3) respectively 4) of Theorem 4.2. Finally, if there are, say, m_0 indices in (τ_i) left with $\tau_i = 1$ and n_0 indices in (ζ_i) left with $\zeta_i = 1$, then we take a block of size $m_0 \times n_0$ as in 1) of Theorem 4.2. Then, by construction and Theorem 4.2, the matrices $\mathcal{H} = G^{-1}A\hat{G}^{-1}A^T$ and $\hat{\mathcal{H}} = \hat{G}^{-1}A^TG^{-1}A$ have the same Jordan canonical form as M and N, respectively. Let $Z, \hat{Z}$ be such that $Z^TGZ = I_m$ and $\hat{Z}^T\hat{G}\hat{Z} = I_n$. Then setting $\hat{A} = Z^T\widetilde{A}\hat{Z}$, we find that $\hat{A}\hat{A}^T = Z^{-1}\mathcal{H}Z$ and $\hat{A}^T\hat{A} = \hat{Z}^{-1}\hat{\mathcal{H}}\hat{Z}$ are symmetric. Thus, there exist orthogonal matrices S and T such that $S\hat{A}\hat{A}^TS^{-1} = M$ and $T\hat{A}^T\hat{A}T^{-1} = N$. (This well-known fact is a direct consequence of Theorem 2.6.) Then $A = S\hat{A}T^{-1}$ satisfies $M = AA^T$ and $N = A^TA$.

Next, assume that $k_1 > 1$. Then 2) guarantees that for each k with $k_{2j-1} > k > k_{2j}$, $j = 1, \dots, \lceil \frac{\nu}{2} \rceil$ we have that $\phi_k \geq 2$. This allows us to modify the sequences (τ_i) and (ζ_i) to (not necessarily monotonically decreasing) sequences $(\widetilde{\tau}_i)$ and $(\widetilde{\zeta}_i)$ such that the number of indices j with $\widetilde{\tau}_j = \widetilde{\zeta}_j = k$ is even for all $k > 1$. In order to avoid too complicated notation, we explain the modification only for the case $\nu \leq 2$. The general case is analogous. Thus, if

$$\begin{aligned}(\tau_i) &= \dots, \underbrace{k_1, \dots, k_1}_{\phi_{k_1}}, \dots, \underbrace{k_1 - 1, \dots, k_1 - 1}_{\phi_{k_1-1}} \dots, \underbrace{k_2 + 1, \dots, k_2 + 1}_{\phi_{k_2+1}} \dots, \underbrace{k_2, \dots, k_2}_{\phi_{k_2}}, \dots, \\ (\zeta_i) &= \dots, \underbrace{k_1, \dots, k_1}_{\phi_{k_1}}, \dots, \underbrace{k_1 - 1, \dots, k_1 - 1}_{\phi_{k_1-1}} \dots, \underbrace{k_2 + 1, \dots, k_2 + 1}_{\phi_{k_2+1}} \dots, \underbrace{k_2, \dots, k_2}_{\phi_{k_2}}, \dots,\end{aligned}$$

then the corresponding parts in the sequences $(\widetilde{\tau}_i)$ and $(\widetilde{\zeta}_i)$ take the forms

$$\begin{aligned}(\widetilde{\tau}_i) &= \dots, \underbrace{k_1, \dots, k_1}_{\phi_{k_1}-1}, \dots, \underbrace{k_1 - 1, \dots, k_1 - 1}_{\phi_{k_1-1}-2} \dots, \underbrace{k_2 + 1, \dots, k_2 + 1}_{\phi_{k_2+1}-2} \dots, \underbrace{k_2, \dots, k_2}_{\phi_{k_2}-1}, \Xi_\tau, \dots, \\ (\widetilde{\zeta}_i) &= \dots, \underbrace{k_1, \dots, k_1}_{\phi_{k_1}-1}, \dots, \underbrace{k_1 - 1, \dots, k_1 - 1}_{\phi_{k_1-1}-2} \dots, \underbrace{k_2 + 1, \dots, k_2 + 1}_{\phi_{k_2+1}-2} \dots, \underbrace{k_2, \dots, k_2}_{\phi_{k_2}-1}, \Xi_\zeta, \dots,\end{aligned}$$

where

$$\begin{array}{rcccccccc} \Xi_\tau &=& k_1, & k_1 - 1, & k_1 - 1, & k_1 - 2, & \dots, & k_2 + 1, & k_2; \\ \Xi_\zeta &=& k_1 - 1, & k_1, & k_1 - 2, & k_1 - 1, & \dots, & k_2, & k_2 + 1. \end{array}$$

When the sequences $(\widetilde{\tau}_i)$ and $(\widetilde{\zeta}_i)$ have been constructed, we can apply the strategy of the previous paragraph to construct A such that $M = AA^T$ and $N = A^TA$. □

Example 4.7 Let

$$M_1 = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}, \quad N_1 = \begin{bmatrix} -1 & i \\ i & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} -1 & i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

i.e., M_1 and N_1 are similar to a Jordan block of size 2×2 associated with zero. Then $(\tau_i^{(1)})_{i\in\mathbb{N}} = (\zeta_i^{(1)})_{i\in\mathbb{N}} = (2, 0, 0, \dots)$ and $(\tau_i^{(2)})_{i\in\mathbb{N}} = (\zeta_i^{(2)})_{i\in\mathbb{N}} = (2, 1, 0, \dots)$ are the sequences as in Theorem 4.5 associated to M_1, N_1 and M_2, N_2, respectively. In both cases, we have $\phi_2 = 1$ which is odd. The sequences associated to M_1 and N_1 do not satisfy condition iii) in Theorem 4.6, while the sequences associated with M_2 and N_2 do. Indeed, there does not exist a matrix A_1 such that $M_1 = A_1 A_1^T$ and $N_1 = A_1^T A_1$, because setting

$$A_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

gives

$$\begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix} = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix} = \begin{bmatrix} -1 & i \\ i & 1 \end{bmatrix}$$

which implies $d = \pm a$. If $d = -a$ then $i = ac - ba = ab - ca$ a contradiction. But $d = a$ implies $a^2 = bc$, because $\det A_1 = \det M_1 = 0$. Moreover, we then have $bc + b^2 = 1 = -c^2 - bc$ which implies $(b+c)^2 = 0$, i.e., $c = -b$, contradicting $a^2 + b^2 = 1 \neq a^2 + c^2$. On the other hand, we have

$$M_2 = AA^T \quad \text{and} \quad N_2 = A^T A, \quad \text{where } A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ i & 1 & 0 \end{bmatrix}.$$

Here, the canonical form for the triple (A, I_3, I_3) is given by

$$\left(\left[\begin{array}{cc|c} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right], \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right], \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \right).$$

5 Condensed forms for G complex symmetric, $\hat{G}$ complex skew-symmetric

In this section we study the canonical forms for the case that G is complex symmetric and $\hat{G}$ complex skew-symmetric. Again, we start with the canonical form for the case that A is quadratic and nonsingular. We cannot directly use our key strategy from the proof of Theorem 4.2 and construct a square root of $\hat{\mathcal{H}}$, because now $\hat{\mathcal{H}}$ is $\hat{G}$-Hamiltonian. A $\hat{G}$-Hamiltonian matrix can neither have a $\hat{G}$-Hamiltonian nor a $\hat{G}$-skew-Hamiltonian square root, because the squares of matrices of such type are always $\hat{G}$-skew-Hamiltonian. Therefore, we will start from the fourth root of the $\hat{G}$-skew-Hamiltonian matrix $\hat{\mathcal{H}}^2$ instead.

Theorem 5.1 *Let $A, G, \hat{G} \in \mathbb{C}^{2n\times 2n}$ be nonsingular and let G be complex symmetric and $\hat{G}$ be complex skew-symmetric. Then there exists nonsingular matrices $X, Y \in \mathbb{C}^{2n\times 2n}$ such that*

$$\begin{aligned} X^T A Y &= \begin{bmatrix} \mathcal{J}_{\xi_1}(\mu_1) & 0 \\ 0 & \mathcal{J}_{\xi_1}(\mu_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\xi_m}(\mu_m) & 0 \\ 0 & \mathcal{J}_{\xi_m}(\mu_m) \end{bmatrix}, \\ X^T G X &= \begin{bmatrix} 0 & R_{\xi_1} \\ R_{\xi_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{\xi_m} \\ R_{\xi_m} & 0 \end{bmatrix}, \\ Y^T \hat{G} Y &= \begin{bmatrix} 0 & R_{\xi_1} \\ -R_{\xi_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{\xi_m} \\ -R_{\xi_m} & 0 \end{bmatrix}, \end{aligned} \tag{5.1}$$

where $\mu_j \in \mathbb{C} \setminus \{0\}$, $\arg \mu_j \in [0, \pi/2)$, and $\xi_j \in \mathbb{N}$ for $j = 1, \ldots, m$. Moreover, for the $\hat{G}$-Hamiltonian matrix $\hat{\mathcal{H}} = \hat{G}^{-1} A^T G^{-1} A$ and the G-skew-symmetric matrix $\mathcal{H} = G^{-1} A \hat{G}^{-1} A^T$

we have that

$$
\begin{aligned}
Y^{-1}\hat{\mathcal{H}}Y &= \begin{bmatrix} -\mathcal{J}_{\xi_1}^2(\mu_1) & 0 \\ 0 & \mathcal{J}_{\xi_1}^2(\mu_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} -\mathcal{J}_{\xi_m}^2(\mu_m) & 0 \\ 0 & \mathcal{J}_{\xi_m}^2(\mu_m) \end{bmatrix}, \\
X^{-1}\mathcal{H}X &= \begin{bmatrix} \mathcal{J}_{\xi_1}^2(\mu_1) & 0 \\ 0 & -\mathcal{J}_{\xi_1}^2(\mu_1) \end{bmatrix}^T \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\xi_m}^2(\mu_m) & 0 \\ 0 & -\mathcal{J}_{\xi_m}^2(\mu_m) \end{bmatrix}^T.
\end{aligned} \tag{5.2}
$$

Proof. By Theorem 2.8, there exists a nonsingular matrix $\mathcal{Y} \in \mathbb{C}^{n\times n}$ such that

$$
\begin{aligned}
\mathcal{Y}^{-1}\hat{\mathcal{H}}\mathcal{Y} &= \begin{bmatrix} \mathcal{J}_{\xi_1}(\lambda_1) & 0 \\ 0 & -\mathcal{J}_{\xi_1}(\lambda_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\xi_m}(\lambda_m) & 0 \\ 0 & -\mathcal{J}_{\xi_m}(\lambda_m) \end{bmatrix}, \\
\mathcal{Y}^T\hat{G}\mathcal{Y} &= \begin{bmatrix} 0 & R_{\xi_1} \\ -R_{\xi_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{\xi_m} \\ -R_{\xi_m} & 0 \end{bmatrix},
\end{aligned}
$$

where $\lambda_j \in \mathbb{C} \setminus \{0\}$, $\arg(\lambda_j) \in [0, \pi)$, and $\xi_j \in \mathbb{N}$ for $j = 1, \ldots, m$. Next construct the matrix $\widetilde{S}$ such that

$$
\mathcal{Y}^{-1}\widetilde{S}\mathcal{Y} = \begin{bmatrix} \mathcal{J}_{\xi_1}(\lambda_1) & 0 \\ 0 & \mathcal{J}_{\xi_1}(\lambda_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\xi_m}(\lambda_m) & 0 \\ 0 & \mathcal{J}_{\xi_m}(\lambda_m) \end{bmatrix}
$$

It is easily verified that $\widetilde{S}$ is $\hat{G}$-skew-Hamiltonian, that it satisfies $\widetilde{S}^2 = \hat{\mathcal{H}}^2$, and that we have $\sigma(\widetilde{S}) \subseteq \{z \in \mathbb{C} \setminus \{0\} : \arg(z) \in [0, \pi)\}$. Thus, by the uniqueness property of Lemma 2.10, we obtain that $\widetilde{S}$ is a polynomial in $\hat{\mathcal{H}}^2$. Moreover, applying Lemma 2.10 once more, we obtain that $\widetilde{S}$ has a unique square root $S \in \mathbb{C}^{n\times n}$ being a polynomial in $\widetilde{S}$ and satisfying $\sigma(S) \subseteq \{z \in \mathbb{C} \setminus \{0\} : \arg(z) \in [0, \pi)\}$, namely

$$
\mathcal{Y}^{-1}S\mathcal{Y} = \begin{bmatrix} \mathcal{J}_{\xi_1}(\lambda_1)^{\frac{1}{2}} & 0 \\ 0 & \mathcal{J}_{\xi_1}(\lambda_1)^{\frac{1}{2}} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\xi_m}(\lambda_m)^{\frac{1}{2}} & 0 \\ 0 & \mathcal{J}_{\xi_m}(\lambda_m)^{\frac{1}{2}} \end{bmatrix}.
$$

In fact, we must have

$$
\sigma(S) \subseteq \{z \in \mathbb{C} \setminus \{0\} : \arg(z) \in [0, \pi/2)\},
$$

because otherwise $\widetilde{S}$ would have eigenvalues λ_j with $\arg(\lambda_j) \in [\pi, 2\pi)$. Let $\mu_j^2 = \lambda_j$ and $\arg(\mu_j) \in [0, \pi/2)$. By Theorem 2.9 we then obtain that there exists a nonsingular matrix $\widetilde{Y} \in \mathbb{C}^{n\times n}$ such that

$$
\begin{aligned}
S_{\text{CF}} := \widetilde{Y}^{-1}S\widetilde{Y} &= \begin{bmatrix} \mathcal{J}_{\xi_1}(\mu_1) & 0 \\ 0 & \mathcal{J}_{\xi_1}(\mu_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\xi_m}(\mu_m) & 0 \\ 0 & \mathcal{J}_{\xi_m}(\mu_m) \end{bmatrix}, \\
\hat{G}_{\text{CF}} := \widetilde{Y}^T\hat{G}\widetilde{Y} &= \begin{bmatrix} 0 & R_{\xi_1} \\ -R_{\xi_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{\xi_m} \\ -R_{\xi_m} & 0 \end{bmatrix}.
\end{aligned}
$$

Moreover, using $G^{-1}A\hat{\mathcal{H}} = \mathcal{H}G^{-1}A$ and the fact that $G^{-1}A$ is nonsingular, we find that $\hat{\mathcal{H}}$ and $\mathcal{H}$ are similar. Thus, by Theorem 2.7 there exists a nonsingular matrix $\widetilde{X} \in \mathbb{C}^{n\times n}$ such that

$$
\begin{aligned}
\mathcal{H}_{\text{CF}} = \widetilde{X}^{-1}\mathcal{H}\widetilde{X} &= \begin{bmatrix} -\mathcal{J}_{\xi_1}^2(\mu_1) & 0 \\ 0 & \mathcal{J}_{\xi_1}^2(\mu_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} -\mathcal{J}_{\xi_m}^2(\mu_m) & 0 \\ 0 & \mathcal{J}_{\xi_m}^2(\mu_m) \end{bmatrix}, \\
G_{\text{CF}} = \widetilde{X}^T G\widetilde{X} &= \begin{bmatrix} 0 & R_{\xi_1} \\ R_{\xi_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{\xi_m} \\ R_{\xi_m} & 0 \end{bmatrix}.
\end{aligned}
$$

Indeed, since $\mathcal{H}$ is similar to $\hat{\mathcal{H}}$, it has the eigenvalues $\lambda_j = \mu_j^2$ with partial multiplicities ξ_j, $j = 1, \ldots, m$. Since the canonical form of G-skew-symmetric matrices in Theorem 2.7 is uniquely determined by the Jordan canonical form, we find that the pairs $(\mathcal{H}, G)$ and $(\mathcal{H}_{\mathrm{CF}}, G_{\mathrm{CF}})$ must have the same canonical form. Observe that S_{CF} is G_{CF}-symmetric, but not a square root of $\mathcal{H}_{\mathrm{CF}}$. Instead, it is easy to check that

$$S_{\mathrm{CF}}(\mathcal{H}_{\mathrm{CF}})^{-1}S_{\mathrm{CF}} = \begin{bmatrix} -I_{\xi_1} & 0 \\ 0 & I_{\xi_1} \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} -I_{\xi_m} & 0 \\ 0 & I_{\xi_m} \end{bmatrix}.$$

Using this identity and setting $X = G^{-1}\widetilde{X}^{-T}$ and $Y = A^{-1}G\widetilde{X}S_{\mathrm{CF}}$, we obtain that

$$\begin{aligned} X^T A Y &= \widetilde{X}^{-1}G^{-1}AA^{-1}G\widetilde{X}S_{\mathrm{CF}} = S_{\mathrm{CF}}, \\ X^T G X &= \widetilde{X}^{-1}G^{-1}GG^{-1}\widetilde{X}^{-T} = (\widetilde{X}^T G\widetilde{X})^{-1} = (G_{\mathrm{CF}})^{-1} = G_{\mathrm{CF}}, \\ Y^T \hat{G} Y &= S_{\mathrm{CF}}^T \widetilde{X}^T G A^{-T}\hat{G}A^{-1}G\widetilde{X}S_{\mathrm{CF}} \\ &= S_{\mathrm{CF}}^T \widetilde{X}^T G\widetilde{X}\widetilde{X}^{-1}\mathcal{H}^{-1}\widetilde{X}S_{\mathrm{CF}} \\ &= S_{\mathrm{CF}}^T G_{\mathrm{CF}}(\mathcal{H}_{\mathrm{CF}})^{-1}S_{\mathrm{CF}} = G_{\mathrm{CF}}S_{\mathrm{CF}}(\mathcal{H}_{\mathrm{CF}})^{-1}S_{\mathrm{CF}} = \hat{G}_{\mathrm{CF}}. \end{aligned}$$

It is now straightforward to check that $Y^{-1}\hat{\mathcal{H}}Y$ and $X^{-1}\mathcal{H}X$ have the claimed forms. Concerning uniqueness, we note that the form (5.1) is already uniquely determined by the Jordan structure of $\hat{\mathcal{H}}$ and by the restriction $\mu_j \in \mathbb{C} \setminus \{0\}$, $\arg \mu_j \in [0, \pi/2)$. □

Theorem 5.2 *Let $A \in \mathbb{C}^{m \times 2n}$, let $G \in \mathbb{C}^{m \times m}$ be complex symmetric and nonsingular and let $\hat{G} \in \mathbb{C}^{2n \times 2n}$ be complex skew-symmetric and nonsingular. Then there exists nonsingular matrices $X \in \mathbb{C}^{m \times m}$ and $Y \in \mathbb{C}^{2n \times 2n}$ such that*

$$\begin{aligned} X^T A Y &= A_c \oplus A_{z,1} \oplus A_{z,2} \oplus A_{z,3} \oplus A_{z,4} \oplus A_{z,5} \oplus A_{z,6}, \\ X^T G X &= G_c \oplus G_{z,1} \oplus G_{z,2} \oplus G_{z,3} \oplus G_{z,4} \oplus G_{z,5} \oplus G_{z,6}, \\ Y^T \hat{G} Y &= \hat{G}_c \oplus \hat{G}_{z,1} \oplus \hat{G}_{z,2} \oplus \hat{G}_{z,3} \oplus \hat{G}_{z,4} \oplus \hat{G}_{z,5} \oplus \hat{G}_{z,6}. \end{aligned} \tag{5.3}$$

Moreover, for the $\hat{G}$-Hamiltonian matrix $\hat{\mathcal{H}} = \hat{G}^{-1}A^T G^{-1}A \in \mathbb{C}^{2n \times 2n}$ and for the G-skew-symmetric matrix $\mathcal{H} = G^{-1}A\hat{G}^{-1}A^T \in \mathbb{C}^{m \times m}$ we have that

$$\begin{aligned} Y^{-1}\hat{\mathcal{H}}Y &= \hat{\mathcal{H}}_c \oplus \hat{\mathcal{H}}_{z,1} \oplus \hat{\mathcal{H}}_{z,2} \oplus \hat{\mathcal{H}}_{z,3} \oplus \hat{\mathcal{H}}_{z,4} \oplus \hat{\mathcal{H}}_{z,5} \oplus \hat{\mathcal{H}}_{z,6}, \\ X^{-1}\mathcal{H}X &= \mathcal{H}_c \oplus \mathcal{H}_{z,1} \oplus \mathcal{H}_{z,2} \oplus \mathcal{H}_{z,3} \oplus \mathcal{H}_{z,4} \oplus \mathcal{H}_{z,5} \oplus \mathcal{H}_{z,6}. \end{aligned}$$

The diagonal blocks in these decompositions have the following forms:

0) *blocks associated with nonzero eigenvalues of $\hat{\mathcal{H}}$ and $\mathcal{H}$:*
   $A_c, G_c, \hat{G}_c$ have the forms as in (5.1) and $\hat{\mathcal{H}}_c, \mathcal{H}_c$ have the forms as in (5.2);

1) *one block corresponding to $2n_0$ Jordan blocks of size 1×1 of $\hat{\mathcal{H}}$ and m_0 Jordan blocks of size 1×1 of $\mathcal{H}$ associated with the eigenvalue zero:*

   $$A_{z,1} = 0_{m_0 \times 2n_0}, \quad G_{z,1} = I_{m_0}, \quad \hat{G}_{z,1} = J_{n_0}, \quad \hat{\mathcal{H}}_{z,1} = 0_{2n_0}, \quad \mathcal{H}_{z,1} = 0_{m_0},$$

   where $m_o, n_o \in \mathbb{N} \cup \{0\}$;

2) *blocks corresponding to a pair of $j \times j$ Jordan blocks of $\hat{\mathcal{H}}$ and $\mathcal{H}$ associated with the eigenvalue zero:*

$$\begin{aligned}
A_{z,2} &= \bigoplus_{i=1}^{\gamma_1} \mathcal{J}_2(0) \oplus \bigoplus_{i=1}^{\gamma_2} \mathcal{J}_4(0) \oplus \cdots \oplus \bigoplus_{i=1}^{\gamma_{2\ell+1}} \mathcal{J}_{4\ell+2}(0) , \\
G_{z,2} &= \bigoplus_{i=1}^{\gamma_1} R_2 \oplus \bigoplus_{i=1}^{\gamma_2} R_4 \oplus \cdots \oplus \bigoplus_{i=1}^{\gamma_{2\ell+1}} R_{4\ell+2} , \\
\hat{G}_{z,2} &= \bigoplus_{i=1}^{\gamma_1} \begin{bmatrix} 0 & R_1 \\ -R_1 & 0 \end{bmatrix} \oplus \bigoplus_{i=1}^{\gamma_2} \begin{bmatrix} 0 & R_2 \\ -R_2 & 0 \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{\gamma_{2\ell+1}} \begin{bmatrix} 0 & R_{2\ell+1} \\ -R_{2\ell+1} & 0 \end{bmatrix} , \\
\hat{\mathcal{H}}_{z,2} &= \bigoplus_{i=1}^{\gamma_1} 0_2 \oplus \bigoplus_{i=1}^{\gamma_2} (-\Sigma_{2,2})\mathcal{J}_4^2(0) \oplus \cdots \oplus \bigoplus_{i=1}^{\gamma_{2\ell+1}} (-\Sigma_{2\ell+1,2\ell+1})\mathcal{J}_{4\ell+2}^2(0) , \\
\mathcal{H}_{z,2} &= \bigoplus_{i=1}^{\gamma_1} 0_2 \oplus \bigoplus_{i=1}^{\gamma_2} \Sigma_{3,1}\mathcal{J}_4^2(0)^T \oplus \cdots \oplus \bigoplus_{i=1}^{\gamma_{2\ell+1}} \Sigma_{2\ell+2,2\ell}\mathcal{J}_{4\ell+2}^2(0)^T ,
\end{aligned}$$

where $\gamma_1, \ldots, \gamma_\ell \in \mathbb{N} \cup \{0\}$; thus, $\hat{\mathcal{H}}_{z,2}$ and $\mathcal{H}_{z,2}$ both have each $2\gamma_j$ Jordan blocks of size $j \times j$ for $j = 1, \ldots, 2\ell + 1$;

3) *blocks corresponding to a $2j \times 2j$ Jordan block of $\hat{\mathcal{H}}$ and a $(2j+1) \times (2j+1)$ Jordan block of $\mathcal{H}$ associated with the eigenvalue zero:*

$$\begin{aligned}
A_{z,3} &= \bigoplus_{i=1}^{m_2} \begin{bmatrix} I_2 \\ 0 \end{bmatrix}_{3\times 2} \oplus \bigoplus_{i=1}^{m_4} \begin{bmatrix} I_4 \\ 0 \end{bmatrix}_{5\times 4} \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\ell}} \begin{bmatrix} I_{2\ell} \\ 0 \end{bmatrix}_{(2\ell+1)\times 2\ell} , \\
G_{z,3} &= \bigoplus_{i=1}^{m_2} R_3 \oplus \bigoplus_{i=1}^{m_4} R_5 \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\ell}} R_{2\ell+1} , \\
\hat{G}_{z,3} &= \bigoplus_{i=1}^{m_2} \begin{bmatrix} 0 & R_1 \\ -R_1 & 0 \end{bmatrix} \oplus \bigoplus_{i=1}^{m_4} \begin{bmatrix} 0 & R_2 \\ -R_2 & 0 \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\ell}} \begin{bmatrix} 0 & R_\ell \\ -R_\ell & 0 \end{bmatrix} , \\
\hat{\mathcal{H}}_{z,3} &= \bigoplus_{i=1}^{m_2} (-\Sigma_{1,1})\mathcal{J}_2(0) \oplus \bigoplus_{i=1}^{m_4} (-\Sigma_{2,2})\mathcal{J}_4(0) \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\ell}} (-\Sigma_{\ell,\ell})\mathcal{J}_{2\ell}(0) , \\
\mathcal{H}_{z,3} &= \bigoplus_{i=1}^{m_2} \Sigma_{2,1}\mathcal{J}_3(0)^T \oplus \bigoplus_{i=1}^{m_4} \Sigma_{3,2}\mathcal{J}_5(0)^T \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\ell}} \Sigma_{\ell+1,\ell}\mathcal{J}_{2\ell+1}(0)^T,
\end{aligned}$$

where $m_2, m_4, \ldots, m_{2\ell} \in \mathbb{N} \cup \{0\}$; thus, $\hat{\mathcal{H}}_{z,3}$ has m_{2j} Jordan blocks of size $2j \times 2j$ and $\mathcal{H}_{z,3}$ has m_{2j} Jordan blocks of size $(2j+1) \times (2j+1)$ for $j = 1, \ldots, \ell$;

4) *blocks corresponding to two $(2j-1) \times (2j-1)$ Jordan blocks of $\hat{\mathcal{H}}$ and two $2j \times 2j$ Jordan*

blocks of $\mathcal{H}$ associated with the eigenvalue zero:

$$
\begin{aligned}
A_{z,4} &= \bigoplus_{i=1}^{m_1} \begin{bmatrix} 0 & I_1 \\ 0 & 0 \\ I_1 & 0 \\ 0 & 0 \end{bmatrix}_{4\times 2} \oplus \bigoplus_{i=1}^{m_3} \begin{bmatrix} 0 & I_3 \\ 0 & 0 \\ I_3 & 0 \\ 0 & 0 \end{bmatrix}_{8\times 6} \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\ell-1}} \begin{bmatrix} 0 & I_{2\ell-1} \\ 0 & 0 \\ I_{2\ell-1} & 0 \\ 0 & 0 \end{bmatrix}_{4\ell\times(4\ell-2)}, \\
G_{z,4} &= \bigoplus_{i=1}^{m_1} R_4 \oplus \bigoplus_{i=1}^{m_3} R_8 \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\ell-1}} R_{4\ell}\ , \\
\hat{G}_{z,4} &= \bigoplus_{i=1}^{m_1} \begin{bmatrix} 0 & R_1 \\ -R_1 & 0 \end{bmatrix} \oplus \bigoplus_{i=1}^{m_3} \begin{bmatrix} 0 & R_3 \\ -R_3 & 0 \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\ell-1}} \begin{bmatrix} 0 & R_{2\ell-1} \\ -R_{2\ell-1} & 0 \end{bmatrix}, \\
\hat{\mathcal{H}}_{z,4} &= \bigoplus_{i=1}^{m_1} 0_2 \oplus \bigoplus_{i=1}^{m_3} \begin{bmatrix} -\mathcal{J}_3(0) & 0 \\ 0 & \mathcal{J}_3(0) \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\ell-1}} \begin{bmatrix} -\mathcal{J}_{2\ell-1}(0) & 0 \\ 0 & \mathcal{J}_{2\ell-1}(0) \end{bmatrix}, \\
\mathcal{H}_{z,4} &= \bigoplus_{i=1}^{m_1} \begin{bmatrix} -\mathcal{J}_2(0) & 0 \\ 0 & \mathcal{J}_2(0) \end{bmatrix}^T \oplus \bigoplus_{i=1}^{m_3} \begin{bmatrix} -\mathcal{J}_4(0) & 0 \\ 0 & \mathcal{J}_4(0) \end{bmatrix}^T \oplus \cdots \oplus \bigoplus_{i=1}^{m_{2\ell-1}} \begin{bmatrix} -\mathcal{J}_{2\ell}(0) & 0 \\ 0 & \mathcal{J}_{2\ell}(0) \end{bmatrix}^T,
\end{aligned}
$$

where $m_1, m_3, \ldots, m_{2\ell-1} \in \mathbb{N} \cup \{0\}$; thus, $\hat{\mathcal{H}}_{z,4}$ has $2m_{2j-1}$ Jordan blocks of the size $(2j-1)\times(2j-1)$ and $\mathcal{H}_{z,4}$ has $2m_{2j-1}$ Jordan blocks of size $2j \times 2j$ for $j = 1, \ldots, \ell$;

5) *blocks corresponding to a $2j \times 2j$ Jordan block of $\hat{\mathcal{H}}$ and a $(2j-1)\times(2j-1)$ Jordan block of $\mathcal{H}$ associated with the eigenvalue zero:*

$$
\begin{aligned}
A_{z,5} &= \bigoplus_{i=1}^{n_1} \begin{bmatrix} 0 & I_1 \end{bmatrix}_{1\times 2} \oplus \bigoplus_{i=1}^{n_3} \begin{bmatrix} 0 & I_3 \end{bmatrix}_{3\times 4} \oplus \cdots \oplus \bigoplus_{i=1}^{n_{2\ell-1}} \begin{bmatrix} 0 & I_{2\ell-1} \end{bmatrix}_{(2\ell-1)\times 2\ell}, \\
G_{z,5} &= \bigoplus_{i=1}^{n_1} R_1 \oplus \bigoplus_{i=1}^{n_3} R_3 \oplus \cdots \oplus \bigoplus_{i=1}^{n_{2\ell-1}} R_{2\ell-1}\ , \\
\hat{G}_{z,5} &= \bigoplus_{i=1}^{n_1} \begin{bmatrix} 0 & R_1 \\ -R_1 & 0 \end{bmatrix} \oplus \bigoplus_{i=1}^{n_3} \begin{bmatrix} 0 & R_2 \\ -R_2 & 0 \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{n_{2\ell-1}} \begin{bmatrix} 0 & R_\ell \\ -R_\ell & 0 \end{bmatrix}, \\
\hat{\mathcal{H}}_{z,5} &= \bigoplus_{i=1}^{n_1} (-\Sigma_{1,1})\mathcal{J}_2(0) \oplus \bigoplus_{i=1}^{n_3} (-\Sigma_{2,2})\mathcal{J}_4(0) \oplus \cdots \oplus \bigoplus_{i=1}^{n_{2\ell-1}} (-\Sigma_{\ell,\ell})\mathcal{J}_{2\ell}(0)\ , \\
\mathcal{H}_{z,5} &= \bigoplus_{i=1}^{n_1} 0_1 \oplus \bigoplus_{i=1}^{n_3} \Sigma_{2,1}\mathcal{J}_3(0)^T \oplus \cdots \oplus \bigoplus_{i=1}^{n_{2\ell-1}} \Sigma_{\ell,\ell-1}\mathcal{J}_{2\ell-1}(0)^T,
\end{aligned}
$$

where $n_1, n_3 \ldots, n_{2\ell-1} \in \mathbb{N} \cup \{0\}$; thus, $\hat{\mathcal{H}}_{z,5}$ has n_{2j-1} Jordan blocks of size $2j \times 2j$ and $\mathcal{H}_{z,5}$ has n_{2j-1} Jordan blocks of size $(2j-1)\times(2j-1)$ for $j = 1, \ldots, \ell$;

6) *blocks corresponding to two $(2j+1)\times(2j+1)$ Jordan blocks of $\hat{\mathcal{H}}$ and two $2j \times 2j$ Jordan blocks of $\mathcal{H}$ associated with the eigenvalue zero:*

$$
\begin{aligned}
A_{z,6} &= \bigoplus_{i=1}^{n_2} \begin{bmatrix} 0 & 0 & 0 & I_2 \\ 0 & I_2 & 0 & 0 \end{bmatrix}_{4\times 6} \oplus \bigoplus_{i=1}^{n_4} \begin{bmatrix} 0 & 0 & 0 & I_4 \\ 0 & I_4 & 0 & 0 \end{bmatrix}_{8\times 10} \oplus \cdots \oplus \bigoplus_{i=1}^{n_{2\ell}} \begin{bmatrix} 0 & 0 & 0 & I_{2\ell} \\ 0 & I_{2\ell} & 0 & 0 \end{bmatrix}_{4\ell\times(4\ell+2)}, \\
G_{z,6} &= \bigoplus_{i=1}^{n_2} R_4 \oplus \bigoplus_{i=1}^{n_4} R_8 \oplus \cdots \oplus \bigoplus_{i=1}^{n_{2\ell}} R_{4\ell}\ , \\
\hat{G}_{z,6} &= \bigoplus_{i=1}^{n_2} \begin{bmatrix} 0 & R_3 \\ -R_3 & 0 \end{bmatrix} \oplus \bigoplus_{i=1}^{n_4} \begin{bmatrix} 0 & R_5 \\ -R_5 & 0 \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{n_{2\ell}} \begin{bmatrix} 0 & R_{2\ell+1} \\ -R_{2\ell+1} & 0 \end{bmatrix}, \\
\hat{\mathcal{H}}_{z,6} &= \bigoplus_{i=1}^{n_2} \begin{bmatrix} -\mathcal{J}_3(0) & 0 \\ 0 & \mathcal{J}_3(0) \end{bmatrix} \oplus \bigoplus_{i=1}^{n_4} \begin{bmatrix} -\mathcal{J}_5(0) & 0 \\ 0 & \mathcal{J}_5(0) \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{n_{2\ell}} \begin{bmatrix} -\mathcal{J}_{2\ell+1}(0) & 0 \\ 0 & \mathcal{J}_{2\ell+1}(0) \end{bmatrix}, \\
\mathcal{H}_{z,6} &= \bigoplus_{i=1}^{n_2} \begin{bmatrix} -\mathcal{J}_2(0) & 0 \\ 0 & \mathcal{J}_2(0) \end{bmatrix}^T \oplus \bigoplus_{i=1}^{n_4} \begin{bmatrix} -\mathcal{J}_4(0) & 0 \\ 0 & \mathcal{J}_4(0) \end{bmatrix}^T \oplus \cdots \oplus \bigoplus_{i=1}^{n_{2\ell}} \begin{bmatrix} -\mathcal{J}_{2\ell}(0) & 0 \\ 0 & \mathcal{J}_{2\ell}(0) \end{bmatrix}^T,
\end{aligned}
$$

where $n_2, n_4, \ldots, n_{2\ell} \in \mathbb{N} \cup \{0\}$*; thus,* $\hat{\mathcal{H}}_{z,6}$ *has* $2n_{2j}$ *Jordan blocks of size* $(2j+1)\times(2j+1)$ *and* $\mathcal{H}_{z,6}$ *has* $2n_{2j}$ *Jordan blocks of size* $2j \times 2j$ *for* $j = 1, \ldots, \ell$*;*

For the eigenvalue zero, the matrices $\hat{\mathcal{H}}$ *and* $\mathcal{H}$ *have* $2\gamma_{2j} + m_{2j} + n_{2j-1}$ *respectively* $2\gamma_{2j} + 2m_{2j-1} + 2n_{2j}$ *Jordan blocks of size* $2j \times 2j$ *for* $j = 1, \ldots, \ell$ *and* $2\gamma_{2j+1} + 2m_{2j+1} + 2n_{2j}$ *respectively* $2\gamma_{2j+1} + m_{2j} + n_{2j+1}$ *Jordan blocks of size* $(2j+1) \times (2j+1)$ *for* $j = 0, \ldots, \ell$*. Here* $m_{2\ell+1} = n_{2\ell+1} = 0$ *and* $2\ell + 1$ *is the smallest odd number that is larger or equal to the maximum of the indices of* $\hat{\mathcal{H}}$ *and* $\mathcal{H}$*. (Here* index *refers to the maximal size of a Jordan block associated with zero.)*

Moreover, the form (4.3) *is unique up to simultaneous block permutation of the blocks in the diagonal blocks of the right hand side of* (4.3)*.*

Proof. The proof is presented in the Appendix. □

6 Canonical forms for $G, \hat{G}$ complex skew-symmetric

In this section we finally treat that case that both G and $\hat{G}$ are complex skew-symmetric.

Theorem 6.1 *Let* $A \in \mathbb{C}^{2n\times 2n}$ *be nonsingular and let* $G, \hat{G} \in \mathbb{C}^{2n\times 2n}$ *be nonsingular and complex skew-symmetric. Then there exists nonsingular matrices* $X, Y \in \mathbb{C}^{2n\times 2n}$ *such that*

$$\begin{aligned}
X^T A Y &= \begin{bmatrix} \mathcal{J}_{\xi_1}(\mu_1) & 0 \\ 0 & \mathcal{J}_{\xi_1}(\mu_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\xi_m}(\mu_m) & 0 \\ 0 & \mathcal{J}_{\xi_m}(\mu_m) \end{bmatrix}, \\
X^T G X &= \begin{bmatrix} 0 & R_{\xi_1} \\ -R_{\xi_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & R_{\xi_m} \\ -R_{\xi_m} & 0 \end{bmatrix}, \\
Y^T \hat{G} Y &= \begin{bmatrix} 0 & -R_{\xi_1} \\ R_{\xi_1} & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & -R_{\xi_m} \\ R_{\xi_m} & 0 \end{bmatrix},
\end{aligned} \tag{6.1}$$

where $\mu_j \in \mathbb{C} \setminus \{0\}$*,* $\arg \mu_j \in [0, \pi)$*, and* $\xi_j \in \mathbb{N}$ *for* $j = 1, \ldots, m$*. Furthermore, for the* $\hat{G}$*-skew-Hamiltonian matrix* $\hat{\mathcal{H}} = \hat{G}^{-1} A^T G^{-1} A$ *and for the* G*-skew-Hamiltonian matrix* $\mathcal{H} = G^{-1} A \hat{G}^{-1} A^T$ *we have that*

$$\begin{aligned}
Y^{-1} \hat{\mathcal{H}} Y &= \begin{bmatrix} \mathcal{J}_{\xi_1}^2(\mu_1) & 0 \\ 0 & \mathcal{J}_{\xi_1}^2(\mu_1) \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\xi_m}^2(\mu_m) & 0 \\ 0 & \mathcal{J}_{\xi_m}^2(\mu_m) \end{bmatrix}, \\
X^{-1} \mathcal{H} X &= \begin{bmatrix} \mathcal{J}_{\xi_1}^2(\mu_1) & 0 \\ 0 & \mathcal{J}_{\xi_1}^2(\mu_1) \end{bmatrix}^T \oplus \cdots \oplus \begin{bmatrix} \mathcal{J}_{\xi_m}^2(\mu_m) & 0 \\ 0 & \mathcal{J}_{\xi_m}^2(\mu_m) \end{bmatrix}^T.
\end{aligned} \tag{6.2}$$

Proof. The proof proceeds completely analogous to the proof of Theorem 4.1. Starting with a skew-Hamiltonian square root S of $\hat{H}$ that is a polynomial in $\hat{H}$ (such a square root exists by Lemma 2.10) and reducing the pair $(S; \hat{G})$ to the canonical form

$$(S_{\text{CF}}, G_{\text{CF}}) = (\widetilde{Y}^{-1} S \widetilde{Y}, \widetilde{Y}^T \hat{G} \widetilde{Y})$$

of Theorem 2.9, we obtain the existence of a transformation matrix $\widetilde{X}$ such that

$$(\widetilde{X}^{-1} \mathcal{H} \widetilde{X}, \widetilde{X}^T G \widetilde{X}) = (S_{\text{CF}}^2, -G_{\text{CF}}).$$

Here, it is used that by Theorem 2.9 the canonical form of all three pairs $(\hat{\mathcal{H}}, \hat{G})$, $(\mathcal{H}, G)$, and $(\mathcal{H}, -G)$ is the same, because $\mathcal{H}$ and $\hat{\mathcal{H}}$ are similar. Then setting $X = G^{-1} \widetilde{X}^{-T}$ and $Y = A^{-1} G \widetilde{X} S_{\text{CF}}$ yields the desired result. □

We mention that the choice of the transformation matrices X, Y in Theorem 6.1 so that $X^TGX = -Y^T\hat{G}Y$ rather than $X^TGX = Y^T\hat{G}Y$ is just a matter of taste. A canonical form (with modified values instead of $\mu_1, \ldots, \mu_m$ in X^TAY) with $X^TGX = Y^T\hat{G}Y$ can be constructed as well, but this would lead to the occurrence of distracting minus signs in the forms for $\mathcal{H}$ and $\hat{\mathcal{H}}$. Therefore, we prefer to represent the canonical form as we did in Theorem 6.1.

Theorem 6.2 *Let $A \in \mathbb{C}^{2m\times 2n}$ and let $G \in \mathbb{C}^{2m\times 2m}$, $\hat{G} \in \mathbb{C}^{2n\times 2n}$ be complex skew-symmetric and nonsingular. Then there exists nonsingular matrices $X \in \mathbb{C}^{2m\times 2m}$ and $Y \in \mathbb{C}^{2n\times 2n}$ such that*

$$\begin{aligned} X^TAY &= A_c \oplus A_{z,1} \oplus A_{z,2} \oplus A_{z,3} \oplus A_{z,4}, \\ X^TGX &= G_c \oplus G_{z,1} \oplus G_{z,2} \oplus G_{z,3} \oplus G_{z,4}, \\ Y^T\hat{G}Y &= \hat{G}_c \oplus \hat{G}_{z,1} \oplus \hat{G}_{z,2} \oplus \hat{G}_{z,3} \oplus \hat{G}_{z,4}. \end{aligned} \tag{6.3}$$

Moreover, for the $\hat{G}$-skew-Hamiltonian matrix $\hat{\mathcal{H}} = \hat{G}^{-1}A^TG^{-1}A \in \mathbb{C}^{2n\times 2n}$ and for the G-skew-Hamiltonian matrix $\mathcal{H} = G^{-1}A\hat{G}^{-1}A^T \in \mathbb{C}^{2m\times 2m}$ we have that

$$\begin{aligned} Y^{-1}\hat{\mathcal{H}}Y &= \hat{\mathcal{H}}_c \oplus \hat{\mathcal{H}}_{z,1} \oplus \hat{\mathcal{H}}_{z,2} \oplus \hat{\mathcal{H}}_{z,3} \oplus \hat{\mathcal{H}}_{z,4}, \\ X^{-1}\mathcal{H}X &= \mathcal{H}_c \oplus \mathcal{H}_{z,1} \oplus \mathcal{H}_{z,2} \oplus \mathcal{H}_{z,3} \oplus \mathcal{H}_{z,4}. \end{aligned}$$

The diagonal blocks in these decompositions have the following forms:

0) *blocks associated with nonzero eigenvalues of $\hat{\mathcal{H}}$ and $\mathcal{H}$:*
   $A_c, G_c, \hat{G}_c$ have the forms as in (6.1) and $\hat{\mathcal{H}}_c, \mathcal{H}_c$ have the forms as in (6.2);

1) *one block corresponding to $2n_0$ Jordan blocks of size 1×1 of $\hat{\mathcal{H}}$ and $2m_0$ Jordan blocks of size 1×1 of $\mathcal{H}$ associated with the eigenvalue zero:*

$$A_{z,1} = 0_{2m_0\times 2n_0}, \quad G_{z,1} = J_{m_0}, \quad \hat{G}_{z,1} = J_{n_0}, \quad \hat{\mathcal{H}}_{z,1} = 0_{2n_0}, \quad \mathcal{H}_{z,1} = 0_{2m_0};$$

2) *blocks corresponding to a pair of $j\times j$ Jordan blocks of $\hat{\mathcal{H}}$ and $\mathcal{H}$ associated with the eigenvalue zero:*

$$\begin{aligned} A_{z,2} &= \bigoplus_{i=1}^{\gamma_1} \mathcal{J}_2(0) \oplus \bigoplus_{i=1}^{\gamma_2} \mathcal{J}_4(0) \oplus \cdots \oplus \bigoplus_{i=1}^{\gamma_\ell} \mathcal{J}_{2\ell}(0), \\ G_{z,2} &= \bigoplus_{i=1}^{\gamma_1} \begin{bmatrix} 0 & R_1 \\ -R_1 & 0 \end{bmatrix} \oplus \bigoplus_{i=1}^{\gamma_2} \begin{bmatrix} 0 & R_2 \\ -R_2 & 0 \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{\gamma_\ell} \begin{bmatrix} 0 & R_\ell \\ -R_\ell & 0 \end{bmatrix}, \\ \hat{G}_{z,2} &= \bigoplus_{i=1}^{\gamma_1} \begin{bmatrix} 0 & R_1 \\ -R_1 & 0 \end{bmatrix} \oplus \bigoplus_{i=1}^{\gamma_2} \begin{bmatrix} 0 & R_2 \\ -R_2 & 0 \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{\gamma_\ell} \begin{bmatrix} 0 & R_\ell \\ -R_\ell & 0 \end{bmatrix}, \\ \hat{\mathcal{H}}_{z,2} &= \bigoplus_{i=1}^{\gamma_1} 0_2 \oplus \bigoplus_{i=1}^{\gamma_2} \hat{\Gamma}_4\mathcal{J}_4^2(0) \oplus \cdots \oplus \bigoplus_{i=1}^{\gamma_\ell} \hat{\Gamma}_{2\ell}\mathcal{J}_{2\ell}^2(0), \\ \mathcal{H}_{z,2} &= \bigoplus_{i=1}^{\gamma_1} 0_2 \oplus \bigoplus_{i=1}^{\gamma_2} \Gamma_4\mathcal{J}_4^2(0)^T \oplus \cdots \oplus \bigoplus_{i=1}^{\gamma_\ell} \Gamma_{2\ell}\mathcal{J}_{2\ell}^2(0)^T, \end{aligned}$$

   where $\gamma_1, \ldots, \gamma_\ell \in \mathbb{N}\cup\{0\}$, $\hat{\Gamma}_{2j} = (-I_{j-1}) \oplus I_1 \oplus (-I_j)$, and $\Gamma_{2j} = (-I_j) \oplus I_1 \oplus (-I_{j-1})$ for $j = 2, \ldots, \ell$; thus, $\hat{\mathcal{H}}_{z,2}$ and $\mathcal{H}_{z,2}$ both have each $2\gamma_j$ Jordan blocks of size $j\times j$ for $j = 1, \ldots, \ell$;

3) *blocks corresponding to two $j \times j$ Jordan blocks of $\hat{\mathcal{H}}$ and two $(j+1)\times(j+1)$ Jordan blocks of $\mathcal{H}$ associated with the eigenvalue zero:*

$$
\begin{aligned}
A_{z,3} &= \bigoplus_{i=1}^{m_1} \begin{bmatrix} 0 & I_1 \\ 0 & 0 \\ I_1 & 0 \\ 0 & 0 \end{bmatrix}_{4\times 2} \oplus \bigoplus_{i=1}^{m_2} \begin{bmatrix} 0 & I_2 \\ 0 & 0 \\ I_2 & 0 \\ 0 & 0 \end{bmatrix}_{6\times 4} \oplus \cdots \oplus \bigoplus_{i=1}^{m_\ell} \begin{bmatrix} 0 & I_{\ell-1} \\ 0 & 0 \\ I_{\ell-1} & 0 \\ 0 & 0 \end{bmatrix}_{2\ell\times(2\ell-2)}, \\
G_{z,3} &= \bigoplus_{i=1}^{m_1} \begin{bmatrix} 0 & R_2 \\ -R_2 & 0 \end{bmatrix} \oplus \bigoplus_{i=1}^{m_2} \begin{bmatrix} 0 & R_3 \\ -R_3 & 0 \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{m_{\ell-1}} \begin{bmatrix} 0 & R_\ell \\ -R_\ell & 0 \end{bmatrix}, \\
\hat{G}_{z,3} &= \bigoplus_{i=1}^{m_1} \begin{bmatrix} 0 & R_1 \\ -R_1 & 0 \end{bmatrix} \oplus \bigoplus_{i=1}^{m_2} \begin{bmatrix} 0 & R_2 \\ -R_2 & 0 \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{m_{\ell-1}} \begin{bmatrix} 0 & R_{\ell-1} \\ -R_{\ell-1} & 0 \end{bmatrix}, \\
\hat{\mathcal{H}}_{z,3} &= \bigoplus_{i=1}^{m_1} 0_2 \oplus \bigoplus_{i=1}^{m_2} \begin{bmatrix} \mathcal{J}_2(0) & 0 \\ 0 & \mathcal{J}_2(0) \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{m_{\ell-1}} \begin{bmatrix} \mathcal{J}_{\ell-1}(0) & 0 \\ 0 & \mathcal{J}_{\ell-1}(0) \end{bmatrix}, \\
\mathcal{H}_{z,3} &= \bigoplus_{i=1}^{m_1} \begin{bmatrix} \mathcal{J}_2(0) & 0 \\ 0 & \mathcal{J}_2(0) \end{bmatrix}^T \oplus \bigoplus_{i=1}^{m_2} \begin{bmatrix} \mathcal{J}_3(0) & 0 \\ 0 & \mathcal{J}_3(0) \end{bmatrix}^T \oplus \cdots \oplus \bigoplus_{i=1}^{m_{\ell-1}} \begin{bmatrix} \mathcal{J}_\ell(0) & 0 \\ 0 & \mathcal{J}_\ell(0) \end{bmatrix}^T,
\end{aligned}
$$

where $m_1, \ldots, m_{\ell-1} \in \mathbb{N} \cup \{0\}$; thus, $\hat{\mathcal{H}}_{z,3}$ has $2m_j$ Jordan blocks of size $j \times j$ and $\mathcal{H}_{z,3}$ has $2m_j$ Jordan blocks of size $(j+1)\times(j+1)$ for $j = 1, \ldots, \ell-1$;

4) *blocks corresponding to two $(j+1)\times(j+1)$ Jordan blocks of $\hat{\mathcal{H}}$ and two $j \times j$ Jordan blocks of $\mathcal{H}$ associated with the eigenvalue zero:*

$$
\begin{aligned}
A_{z,4} &= \bigoplus_{i=1}^{n_1} \begin{bmatrix} 0 & 0 & 0 & I_1 \\ 0 & I_1 & 0 & 0 \end{bmatrix}_{2\times 4} \oplus \bigoplus_{i=1}^{n_2} \begin{bmatrix} 0 & 0 & 0 & I_2 \\ 0 & I_2 & 0 & 0 \end{bmatrix}_{4\times 6} \oplus \cdots \oplus \bigoplus_{i=1}^{n_{\ell-1}} \begin{bmatrix} 0 & 0 & 0 & I_{\ell-1} \\ 0 & I_{\ell-1} & 0 & 0 \end{bmatrix}_{(2\ell-2)\times 2\ell}, \\
G_{z,4} &= \bigoplus_{i=1}^{n_1} \begin{bmatrix} 0 & R_1 \\ -R_1 & 0 \end{bmatrix} \oplus \bigoplus_{i=1}^{n_2} \begin{bmatrix} 0 & R_2 \\ -R_2 & 0 \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{n_{\ell-1}} \begin{bmatrix} 0 & R_{\ell-1} \\ -R_{\ell-1} & 0 \end{bmatrix} \\
\hat{G}_{z,4} &= \bigoplus_{i=1}^{n_1} \begin{bmatrix} 0 & R_2 \\ -R_2 & 0 \end{bmatrix} \oplus \bigoplus_{i=1}^{n_2} \begin{bmatrix} 0 & R_3 \\ -R_3 & 0 \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{n_{\ell-1}} \begin{bmatrix} 0 & R_\ell \\ -R_\ell & 0 \end{bmatrix}, \\
\hat{\mathcal{H}}_{z,4} &= \bigoplus_{i=1}^{n_1} \begin{bmatrix} \mathcal{J}_2(0) & 0 \\ 0 & \mathcal{J}_2(0) \end{bmatrix} \oplus \bigoplus_{i=1}^{n_2} \begin{bmatrix} \mathcal{J}_3(0) & 0 \\ 0 & \mathcal{J}_3(0) \end{bmatrix} \oplus \cdots \oplus \bigoplus_{i=1}^{n_{\ell-1}} \begin{bmatrix} \mathcal{J}_\ell(0) & 0 \\ 0 & \mathcal{J}_\ell(0) \end{bmatrix}, \\
\mathcal{H}_{z,4} &= \bigoplus_{i=1}^{n_1} \begin{bmatrix} \mathcal{J}_1(0) & 0 \\ 0 & \mathcal{J}_1(0) \end{bmatrix}^T \oplus \bigoplus_{i=1}^{n_2} \begin{bmatrix} \mathcal{J}_2(0) & 0 \\ 0 & \mathcal{J}_2(0) \end{bmatrix}^T \oplus \cdots \oplus \bigoplus_{i=1}^{n_{\ell-1}} \begin{bmatrix} \mathcal{J}_{\ell-1}(0) & 0 \\ 0 & \mathcal{J}_{\ell-1}(0) \end{bmatrix}^T,
\end{aligned}
$$

where $n_1, \ldots, n_{\ell-1} \in \mathbb{N} \cup \{0\}$; thus, $\hat{\mathcal{H}}_{z,4}$ has $2n_j$ Jordan blocks of size $(j+1)\times(j+1)$ and $\mathcal{H}_{z,4}$ has $2n_j$ Jordan blocks of size $j \times j$ for $j = 1, \ldots, \ell-1$;

Then for the eigenvalue zero, the matrices $\hat{\mathcal{H}}$ and $\mathcal{H}$ have $2\gamma_j + 2m_j + 2n_{j-1}$ respectively $2\gamma_j + 2m_{j-1} + 2n_j$ Jordan blocks of size $j \times j$ for $j = 1, \ldots, \ell$. Here ℓ is the maximum of the indices of $\hat{\mathcal{H}}$ and $\mathcal{H}$. (Here, index *refers to the maximal size of a Jordan block associated with the eigenvalue zero.)*

Moreover, the form (6.3) is unique up to simultaneous block permutation of the blocks in the diagonal blocks of the right hand side of (6.3).

Proof. The proof is presented in the Appendix. □

7 Conclusion

We have presented canonical forms for matrix triples $(A, G, \hat{G})$ where $G, \hat{G}$ are complex symmetric or complex skew-symmetric and nonsingular. The canonical form for A can be interpreted as a variant of the singular value decomposition, because the form also displays the Jordan canonical forms of the structured matrices $\hat{\mathcal{H}} = \hat{G}^{-1}A^TG^{-1}A$ and $\mathcal{H} = G^{-1}A\hat{G}^{-1}A^T$.

Acknowledgement

We thank Leiba Rodman for some valuable comments and in particular for pointing us into the direction of Theorem 4.6.

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Appendix: Proofs of the main theorems

In the appendix, we present a constructive and recursive proof of Theorem 4.2. Then, we explain the necessary changes to be made in the proof to obtain the proofs of Theorems 5.2 and 6.2.

Proof of Theorem 4.2

The proof proceeds in four well-separated steps. First, we present a reduction towards a staircase-like form by repeatedly applying Proposition 3.3. In the second step, we further reduce this staircase-like form towards a form that can be considered as a canonical form. In the third step, we show how single Jordan blocks can be extracted from the form. Finally, uniqueness is proved in the fourth step.

Step 1) Reduction to a stair-case-like form

Applying appropriate congruence transformations to G and $\hat{G}$ otherwise, we may assume that $G = I_m$ and $\hat{G} = I_n$. Let

$$A = B_1 C_1^T$$

be a full rank factorization of A, i.e., $B_1 \in \mathbb{C}^{m \times r}$, $C_1 \in \mathbb{C}^{n \times r}$, $\operatorname{rank} B_1 = \operatorname{rank} C_1 = r$. Applying Proposition 3.3 to B_1 and C_1, respectively, we can determine nonsingular matrices $X_1 \in \mathbb{C}^{m \times m}$ and $Y_1 \in \mathbb{C}^{n \times n}$ such that

$$\begin{aligned} X_1^T B_1 &= \begin{bmatrix} 0 \\ 0 \\ B_{10} \end{bmatrix} \begin{matrix} \scriptstyle \pi_0 \\ \scriptstyle \delta_1 \\ \scriptstyle r \end{matrix}, \quad X_1^T X_1 = I_{\pi_0} \oplus \begin{bmatrix} 0 & 0 & I_{\delta_1} \\ 0 & I_{p_1} & 0 \\ I_{\delta_1} & 0 & 0 \end{bmatrix}, \\ Y_1^T C_1 &= \begin{bmatrix} 0 \\ 0 \\ C_{10} \end{bmatrix} \begin{matrix} \scriptstyle \hat{\pi}_0 \\ \scriptstyle \hat{\delta}_1 \\ \scriptstyle r \end{matrix}, \quad Y_1^T Y_1 = I_{\hat{\pi}_0} \oplus \begin{bmatrix} 0 & 0 & I_{\hat{\delta}_1} \\ 0 & I_{\hat{p}_1} & 0 \\ I_{\hat{\delta}_1} & 0 & 0 \end{bmatrix}, \end{aligned}$$

where $B_{10}, C_{10} \in \mathbb{C}^{r\times r}$ are both invertible, $p_1, \delta_1, \hat{p}_1, \hat{\delta}_1 \geq 0$, and

$$p_1 + \delta_1 = \hat{p}_1 + \hat{\delta}_1 = r.$$

Partition

$$B_{10}C_{10}^T = \begin{array}{c} \\ p_1 \\ \delta_1 \end{array}\!\!\begin{array}{c} \begin{array}{cc} \hat{p}_1 & \hat{\delta}_1 \end{array} \\ \begin{bmatrix} A_{33} & A_{34} \\ A_{43} & A_{44} \end{bmatrix} \end{array},$$

then

$$X_1^T A Y_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix}, \quad X_1^T X_1 = \begin{bmatrix} I_{\pi_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\delta_1} \\ 0 & 0 & I_{p_1} & 0 \\ 0 & I_{\delta_1} & 0 & 0 \end{bmatrix},$$

$$Y_1^T Y_1 = \begin{bmatrix} I_{\hat{\pi}_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\hat{\delta}_1} \\ 0 & 0 & I_{\hat{p}_1} & 0 \\ 0 & I_{\hat{\delta}_1} & 0 & 0 \end{bmatrix},$$

Applying the same procedure to the triple $(A_{33}, I_{p_1}, I_{\hat{p}_1})$, we can construct nonsingular matrices $\widetilde{X}_2, \widetilde{Y}_2$ such that

$$\widetilde{X}_2^T A_{33} \widetilde{Y}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_{55} & A_{56} \\ 0 & 0 & A_{65} & A_{66} \end{bmatrix}, \quad \widetilde{X}_2^T \widetilde{X}_2 = \begin{bmatrix} I_{\pi_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\delta_2} \\ 0 & 0 & I_{p_2} & 0 \\ 0 & I_{\delta_2} & 0 & 0 \end{bmatrix},$$

$$\widetilde{Y}_2^T \widetilde{Y}_2 = \begin{bmatrix} I_{\hat{\pi}_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\hat{\delta}_2} \\ 0 & 0 & I_{\hat{p}_2} & 0 \\ 0 & I_{\hat{\delta}_2} & 0 & 0 \end{bmatrix},$$

where $p_2, \delta_2, \hat{p}_2, \hat{\delta}_2 \geq 0$, $A_{66} \in \mathbb{F}^{\delta_2 \times \hat{\delta}_2}$, $A_{56} \in \mathbb{F}^{p_2 \times \hat{\delta}_2}$, $A_{65} \in \mathbb{F}^{\delta_2 \times \hat{p}_2}$, $A_{55} \in \mathbb{F}^{p_2 \times \hat{p}_2}$, and $p_2 + \delta_2 = \hat{p}_2 + \hat{\delta}_2 = \operatorname{rank} A_{33}$, and where the matrix

$$\begin{bmatrix} A_{55} & A_{56} \\ A_{65} & A_{66} \end{bmatrix} \in \mathbb{F}^{(p_2+\delta_2)\times(p_2+\delta_2)}$$

is nonsingular. Letting

$$X_2 = X_1(I_{\pi_0+\delta_1} \oplus \widetilde{X}_2 \oplus I_{\delta_1}), \quad Y_2 = Y_1(I_{\hat{\pi}_0+\hat{\delta}_1} \oplus \widetilde{Y}_2 \oplus I_{\hat{\delta}_1}),$$

we then have

$$X_2^T A Y_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{37} \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{47} \\ 0 & 0 & 0 & 0 & A_{55} & A_{56} & A_{57} \\ 0 & 0 & 0 & 0 & A_{65} & A_{66} & A_{67} \\ 0 & 0 & A_{73} & A_{74} & A_{75} & A_{76} & A_{77} \end{bmatrix}, \quad X_2^T X_2 = \begin{bmatrix} I_{\pi_0} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{\delta_1} \\ 0 & 0 & I_{\pi_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{\delta_2} & 0 \\ 0 & 0 & 0 & 0 & I_{p_2} & 0 & 0 \\ 0 & 0 & 0 & I_{\delta_2} & 0 & 0 & 0 \\ 0 & I_{\delta_1} & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Y_2^T Y_2 = \begin{bmatrix} I_{\hat{\pi}_0} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{\hat{\delta}_1} \\ 0 & 0 & I_{\hat{\pi}_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{\hat{\delta}_2} & 0 \\ 0 & 0 & 0 & 0 & I_{\hat{p}_2} & 0 & 0 \\ 0 & 0 & 0 & I_{\hat{\delta}_2} & 0 & 0 & 0 \\ 0 & I_{\hat{\delta}_1} & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where the matrix $X_2^T A Y_2$ has been partitioned conformably with $X_2^T X_2$ (row-wise) and $Y_2^T Y_2$ (column-wise). The submatrix of $X_2^T A Y_2$ that is obtained by deleting the leading two rows and columns is then nonsingular, because it is equivalent to $B_{10} C_{10}^T$. Thus, $\left[\begin{smallmatrix} A_{37} \\ A_{47} \end{smallmatrix}\right]$ has full row rank and $[A_{73} \;\; A_{74}]$ has full column rank.

We can repeat the procedure for the triple $(A_{55}, I_{p_2}, I_{\hat{p}_2})$ which finally yields nonsingular matrices X_3 and Y_3 such that (after renaming some blocks in A and using the canonical notation corresponding to the notation in the previous step), we have

$$X_3^T A Y_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{3,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{4,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{5,9} & A_{5,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{6,9} & A_{6,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{7,7} & A_{7,8} & A_{7,9} & A_{7,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{8,7} & A_{8,8} & A_{8,9} & A_{8,10} \\ 0 & 0 & 0 & 0 & A_{9,5} & A_{9,6} & A_{9,7} & A_{9,8} & A_{9,9} & A_{9,10} \\ 0 & 0 & A_{10,3} & A_{10,4} & A_{10,5} & A_{10,6} & A_{10,7} & A_{10,8} & A_{10,9} & A_{10,10} \end{bmatrix}, \tag{7.1}$$

$$X_3^T X_3 = \begin{bmatrix} I_{\pi_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\delta_1} \\ 0 & 0 & I_{\pi_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\delta_2} & 0 \\ 0 & 0 & 0 & 0 & I_{\pi_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\delta_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{p_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{\delta_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\delta_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{\delta_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{7.2}$$

$$Y_3^T Y_3 = \begin{bmatrix} I_{\hat{\pi}_0}, & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\hat{\delta}_1} \\ 0 & 0 & I_{\hat{\pi}_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\hat{\delta}_2} & 0 \\ 0 & 0 & 0 & 0 & I_{\hat{\pi}_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\hat{\delta}_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{\hat{p}_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{\hat{\delta}_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\hat{\delta}_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{\hat{\delta}_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{7.3}$$

where $[A_{10,3} \;\; A_{10,4}]$ and $[A_{9,5} \;\; A_{9,6}]$ have full column rank,

$$\begin{bmatrix} A_{3,10} \\ A_{4,10} \end{bmatrix} \text{ and } \begin{bmatrix} A_{5,9} \\ A_{6,9} \end{bmatrix} \text{ have full row rank, and } \begin{bmatrix} A_{77} & A_{78} \\ A_{87} & A_{88} \end{bmatrix} \text{ is nonsingular.}$$

Continuing recursively, the process clearly has to stagnate after finitely many steps. Using the canonical notation corresponding to the notation in the first two steps of the process, we find that stagnation occurs after the ℓth step either when $A_{2\ell+1,2\ell+1}$ is nonsingular or when $p_\ell = \hat{p}_\ell = 0$. In both cases we obviously have that $p_\ell = \hat{p}_\ell$, and we end up with a nonsingular matrix

$$\begin{bmatrix} A_{2\ell+1,2\ell+1} & A_{2\ell+1,2\ell+2} \\ A_{2\ell+2,2\ell+1} & A_{2\ell+2,2\ell+2} \end{bmatrix} \in \mathbb{F}^{(p_\ell+\delta_\ell)\times(\hat{p}_\ell+\hat{\delta}_\ell)}, \tag{7.4}$$

full row rank matrices

$$\begin{bmatrix} A_{2k+1,3\ell+2-k} \\ A_{2k+2,3\ell+2-k} \end{bmatrix} \in \mathbb{F}^{(\pi_k+\delta_{k+1})\times\hat{\delta}_k}, \quad k = 1, \ldots, \ell-1,$$

and full column rank matrices $[A_{3\ell+2-k,2k+1} \;\; A_{3\ell+2-k,2k+2}] \in \mathbb{F}^{\delta_k \times (\hat{\pi}_k + \hat{\delta}_{k+1})}$ for $k = 1, \ldots, \ell-1$. Also, we have

$$\delta_\ell = \hat{\delta}_\ell, \tag{7.5}$$

because $p_\ell + \delta_\ell = \hat{p}_\ell + \hat{\delta}_\ell$. Finally, we obtain that due to the full rank properties, we have that

$$\delta_{k-1} \geq \hat{\pi}_{k-1} + \hat{\delta}_k, \quad \hat{\delta}_{k-1} \geq \pi_{k-1} + \delta_k \tag{7.6}$$

for $k = 2, \ldots, \ell$. On the other hand, the nonsingularity of the submatrices in (7.4) implies that

$$p_k + \delta_k = \hat{p}_k + \hat{\delta}_k \tag{7.7}$$

for $k = 1, 2, \ldots, \ell-1$. We also have

$$\begin{aligned} p_{k-1} &= \pi_{k-1} + 2\delta_k + p_k, \\ \hat{p}_{k-1} &= \hat{\pi}_{k-1} + 2\hat{\delta}_k + \hat{p}_k, \end{aligned}$$

for $k = 2, \ldots, l$. The latter two equations can be rewritten as

$$\begin{aligned} p_{k-1} + \delta_{k-1} &= \pi_{k-1} + \delta_k + \delta_{k-1} + (p_k + \delta_k), \\ \hat{p}_{k-1} + \hat{\delta}_{k-1} &= \hat{\pi}_{k-1} + \hat{\delta}_k + \hat{\delta}_{k-1} + (\hat{p}_k + \hat{\delta}_k). \end{aligned}$$

By using (7.7) we then obtain

$$\pi_{k-1} + \delta_k + \delta_{k-1} = \hat{\pi}_{k-1} + \hat{\delta}_k + \hat{\delta}_{k-1},$$

or, equivalently,

$$\hat{\delta}_{k-1} - \pi_{k-1} - \delta_k = \delta_{k-1} - \hat{\pi}_{k-1} - \hat{\delta}_k \geq 0 \tag{7.8}$$

for $k = 2, \ldots, \ell$, where the nonnegativity follows from (7.6).

Step 2) Further reduction of the staircase form

We now isolate the nonsingular block $A_{2\ell+1,2\ell+1}$ from the other blocks and compress the remaining part of $X_\ell^T AY_\ell$ to more condensed form. We set $\pi_\ell = p_\ell, \hat{\pi}_\ell = \hat{p}_\ell$ and

$$m_k := \begin{cases} \pi_k & \text{if } k \text{ is even} \\ \hat{\pi}_k & \text{if } k \text{ is odd} \end{cases}, \quad n_k := \begin{cases} \pi_k & \text{if } k \text{ is odd} \\ \hat{\pi}_k & \text{if } k \text{ is even} \end{cases}$$

for $k = 0, \ldots, \ell$. Moreover, (using (7.5) and (7.8)), we define $\gamma_\ell := \delta_\ell = \hat{\delta}_\ell$ and

$$\gamma_k := \hat{\delta}_k - \pi_k - \delta_{k+1} = \delta_k - \hat{\pi}_k - \hat{\delta}_{k+1}, \quad k = 1, \ldots, \ell - 1.$$

For the sake of readability of the paper, we will not carry out the proof for the general case, but we will illustrate the procedure for the special case that $\ell = 3$, where we have the matrices as in (7.1)– (7.3). The general case proceeds completely analogous, but the tedious details are left to the reader.

If not void then $A_{7,7}$ in $X_3^T AY_3$ in (7.1) is nonsingular, and hence, we can annihilate $A_{7,8}$ by post-multiplying $X_3^T AY_3$ with the matrix

$$Z_1 := I_{n_0} \oplus I_{\hat{\delta}_1} \oplus I_{m_1} \oplus I_{\hat{\delta}_2} \oplus I_{n_2} \oplus I_{\hat{\delta}_3} \oplus \begin{bmatrix} I & -A_{7,7}^{-1}A_{7,8} \\ 0 & I \end{bmatrix} \oplus I_{\hat{\delta}_2} \oplus I_{\hat{\delta}_1}.$$

Correspondingly updating $Y_3^T Y_3$ this leads to a fill-in in the $(7,8)$ and $(8,7)$ block positions in $Z_1^T Y_3^T Y_3 Z_1$ given by $-A_{7,7}^{-1}A_{7,8}$ and $-A_{7,8}^T A_{7,7}^{-T}$, respectively. We can annihilate these two fill-ins by using the $(8,6)$ block entry $I_{\hat{\delta}_3}$ as a pivot, i.e., by applying a congruence transformation to $Z_1^T Y_3^T Y_3 Z_1$ with

$$Z_2 = I_{n_0} \oplus I_{\hat{\delta}_1} \oplus I_{m_1} \oplus I_{\hat{\delta}_2} \oplus I_{n_2} \oplus \begin{bmatrix} I & A_{7,8}^T A_{7,7}^{-T} \\ 0 & I \end{bmatrix} \oplus I_{\hat{\delta}_3} \oplus I_{\hat{\delta}_2} \oplus I_{\hat{\delta}_1}.$$

It is then easy to check that $Z_2^T Z_1^T Y_3^T Y_3 Z_1 Z_2 = Y_3^T Y_3$ and that the correspondingly updated matrix $X_3^T AY_3 Z_1 Z_2$ has no further fill-ins. Finally, we update $Y_3 \leftarrow Y_3 Z_1 Z_2$.

Similarly, we can annihilate $A_{8,7}$ by working on the rows of $X_3^T AY_3$ and applying congruence transformations to $X_3^T X_3$. Then, we can proceed and annihilate the blocks $A_{7,9}$, $A_{9,7}$, $A_{7,10}$, and $A_{10,7}$ in $X_3^T AY_3$. Since originally the matrix

$$\begin{bmatrix} A_{7,7} & A_{7,8} \\ A_{8,7} & A_{8,8} \end{bmatrix}$$

is nonsingular, we find that after the above reductions the updated block $A_{8,8}$ is nonsingular (or even void). With $A_{8,8}$ as the pivot, we can then annihilate $A_{8,9}$, $A_{9,8}$, $A_{8,10}$, $A_{10,8}$ and recover $X_3^T X_3$ and $Y_3^T Y_3$. Observe that this does not change the zero blocks in $X_3^T AY_3$. Finally post-multiplying $X_3^T AY_3$ with the matrix

$$Z_3 = I_{n_0} \oplus I_{\hat{\delta}_1} \oplus I_{m_1} \oplus I_{\hat{\delta}_2} \oplus I_{n_2} \oplus A_{8,8}^T \oplus I_{\pi_3} \oplus A_{8,8}^{-1} \oplus I_{\hat{\delta}_2} \oplus I_{\hat{\delta}_1},$$

(and updating $Y_3 \leftarrow Y_3 Z_3$) we then obtain

$$X_3^T A Y_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{3,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{4,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{5,9} & A_{5,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{6,9} & A_{6,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{7,7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\delta_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{9,5} & A_{9,6} & 0 & 0 & A_{9,9} & A_{9,10} \\ 0 & 0 & A_{10,3} & A_{10,4} & A_{10,5} & A_{10,6} & 0 & 0 & A_{10,9} & A_{10,10} \end{bmatrix},$$

while $X_3^T X_3$ and $Y_3^T Y_3$ are as in (7.2) and (7.3). (Indeed, observe that the congruence transformation with Z_3 leaves $Y_3^T Y_3$ invariant.) Since the original block $[A_{9,5}\ \ A_{9,6}]$ has full column rank, it easily follows that the corresponding updated entry

$$\begin{bmatrix} A_{9,5} & A_{9,6} \end{bmatrix} \leftarrow \begin{bmatrix} A_{9,5} & A_{9,6} A_{8,8}^T \end{bmatrix}$$

has full column rank as well. Then there exists a nonsingular matrix W_1 such that

$$\begin{bmatrix} A_{9,5} & A_{9,6} \end{bmatrix} \leftarrow W_1^T \begin{bmatrix} A_{9,5} & A_{9,6} \end{bmatrix} = \begin{bmatrix} I_{n_2} & 0 \\ 0 & I_{\hat{\delta}_3} \\ 0 & 0 \end{bmatrix}. \tag{7.9}$$

Transforming then $X_3^T A Y_3$ and $X_3^T X_3$ with a pre-multiplication and congruence transformation, respectively, with a block diagonal matrix having W_1^{-1} in the $(4,4)$-block position and W_1^T in the $(9,9)$-block position, we obtain the desired update in the block $[A_{9,5}\ \ A_{9,6}]$ while $X_3^T X_3$ and zero block-structure of $X_3^T A Y_3$ are invariant under that transformation. We then continue by taking this updated block $[A_{9,5}\ \ A_{9,6}]$ as a pivot to annihilate $[A_{10,5}\ \ A_{10,6}]$. Again, this can be done without changing $X_3^T X_3$.

Similarly, due to a full row rank argument, there exists a nonsingular matrix W_2 such that

$$\begin{bmatrix} A_{5,9} \\ A_{6,9} \end{bmatrix} := \begin{bmatrix} A_{5,9} \\ A_{6,9} \end{bmatrix} W_2 = \begin{bmatrix} I_{m_2} & 0 & 0 \\ 0 & I_{\delta_3} & 0 \end{bmatrix}. \tag{7.10}$$

and applying appropriate transformation matrices, the corresponding change in $X_3^T A Y_3$ can be made without changing $Y_3^T Y_3$. Then, $A_{5,10}$ and $A_{6,10}$ can be annihilated.

Also, we use the pivots $\left[\begin{smallmatrix} A_{5,9} \\ A_{6,9} \end{smallmatrix}\right]$ and $\begin{bmatrix} A_{9,5} & A_{9,6} \end{bmatrix}$, respectively, to annihilate the leading $m_2 + \delta_3$ columns of $A_{9,9}$ and $A_{10,9}$, and the leading $n_2 + \hat{\delta}_3$ rows of $A_{9,9}$ and $A_{9,10}$. So these three blocks become

$$A_{9,9} \leftarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \widetilde{A}_{9,9} \end{bmatrix}, \quad A_{9,10} \leftarrow \begin{bmatrix} 0 \\ 0 \\ \widetilde{A}_{9,10} \end{bmatrix}, \quad A_{10,9} \leftarrow \begin{bmatrix} 0 & 0 & \widetilde{A}_{10,9} \end{bmatrix},$$

where $\widetilde{A}_{9,9} \in \mathbb{F}^{\gamma_2 \times \gamma_2}$, $\widetilde{A}_{9,10} \in \mathbb{F}^{\gamma_2 \times \hat{\delta}_1}$, $\widetilde{A}_{10,9} \in \mathbb{F}^{\delta_1 \times \gamma_2}$. Since originally the submatrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & A_{5,9} \\ 0 & 0 & 0 & 0 & A_{6,9} \\ 0 & 0 & A_{7,7} & A_{7,8} & A_{7,9} \\ 0 & 0 & A_{8,7} & A_{8,8} & A_{8,9} \\ A_{9,5} & A_{9,6} & A_{9,7} & A_{9,8} & A_{9,9} \end{bmatrix}$$

was nonsingular, we have that $\widetilde{A}_{9,9}$ is nonsingular. We then use $\widetilde{A}_{9,9}$ as pivot block to annihilate $\widetilde{A}_{9,10}$ and $\widetilde{A}_{10,9}$, and transform $\widetilde{A}_{9,9}$ to I_{γ_2}.

In a similar way we can perform the reductions

$$\begin{bmatrix} A_{3,10} \\ A_{4,10} \end{bmatrix} \leftarrow \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{\delta_2} & 0 \end{bmatrix}, \quad \begin{bmatrix} A_{10,3} & A_{10,4} \end{bmatrix} \leftarrow \begin{bmatrix} I_{m_1} & 0 \\ 0 & I_{\hat{\delta}_2} \\ 0 & 0 \end{bmatrix},$$

and use them as pivots to reduce $A_{10,10}$ to

$$A_{10,10} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \widetilde{A}_{10,10} \end{bmatrix},$$

where $\widetilde{A}_{10,10} \in \mathbb{F}^{\gamma_1 \times \gamma_1}$, and finally transform $\widetilde{A}_{10,10}$ to I_{γ_1}. After all this, the matrix $X_3^T AY_3$ has the form

$$X_3^T AY_3 = \left[\begin{array}{cccccccc|ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{n_1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\delta_2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{m_2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\gamma_3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_{7,7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\gamma_3} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & I_{n_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_{\gamma_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\gamma_2} & 0 & 0 & 0 \\
\hline
0 & 0 & I_{m_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{\hat{\delta}_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{\gamma_1}
\end{array}\right],$$

while $X_3^T X_3$ and $Y_3^T Y_3$ are still as in (7.2) and (7.3). We partition

$$\begin{aligned} I_{\delta_1} &= I_{m_1} \oplus I_{m_2} \oplus I_{\gamma_3} \oplus I_{\gamma_2} \oplus I_{\gamma_1}, & I_{\delta_2} &= I_{n_2} \oplus I_{\gamma_3} \oplus I_{\gamma_2}, \\ I_{\hat{\delta}_1} &= I_{n_1} \oplus I_{n_2} \oplus I_{\gamma_3} \oplus I_{\gamma_2} \oplus I_{\gamma_1}, & I_{\hat{\delta}_2} &= I_{m_2} \oplus I_{\gamma_3} \oplus I_{\gamma_2}, \end{aligned}$$

and replace I_{δ_1}, I_{δ_2}, $I_{\hat{\delta}_1}$, and $I_{\hat{\delta}_2}$ in the matrix triple with these partitions. We then get $X_3^T AY_3$, $X_3^T X_3$, and $Y_3^T Y_3$ partitioned in 22 block rows and columns. Let P_R be the block permutation that re-arranges the block columns of $X_3^T AY_3$ in the order

$$13, 1, 6, 22, 5, 10, 17, 21, 4, 9, 12, 14, 16, 20, 2, 7, 18, 3, 8, 11, 15, 19.$$

Let P_L be another block permutation such that P_L^T re-arranges the block rows of $X_3^T AY_3$ in the same order. Set

$$\widetilde{X} := X_3 P_L, \quad \widetilde{Y} := Y_3 P_R.$$

Then we obtain that

$$\begin{aligned} \widetilde{X}^T A \widetilde{Y} &= \mathcal{A}_{ns} \oplus \mathcal{A}_0 \oplus (\mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3) \oplus (\mathcal{A}_{1,2} \oplus \mathcal{A}_{2,3}), \\ \widetilde{X}^T \widetilde{X} &= \mathcal{G}_{ns} \oplus \mathcal{G}_0 \oplus (\mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \mathcal{G}_3) \oplus (\mathcal{G}_{1,2} \oplus \mathcal{G}_{2,3}), \\ \widetilde{Y}^T \widetilde{Y} &= \hat{\mathcal{G}}_{ns} \oplus \hat{\mathcal{G}}_0 \oplus (\hat{\mathcal{G}}_1 \oplus \hat{\mathcal{G}}_2 \oplus \hat{\mathcal{G}}_3) \oplus (\hat{\mathcal{G}}_{1,2} \oplus \hat{\mathcal{G}}_{2,3}), \end{aligned}$$

where

$$\mathcal{A}_{ns} = A_{2\ell+1,2\ell+1}, \quad \mathcal{G}_{ns} = I_{\pi_\ell} = I_{n_\ell}, \quad \hat{\mathcal{G}}_{ns} = I_{\hat{\pi}_\ell} = I_{m_\ell}, \quad \ell = 3 \tag{7.11}$$

$$\mathcal{A}_0 = 0_{m_0 \times n_0}, \quad \mathcal{G}_0 = I_{\pi_0} = I_{m_0}, \quad \hat{\mathcal{G}}_0 = I_{\hat{\pi}_0} = I_{n_0}, \tag{7.12}$$

$$\mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3 = \begin{bmatrix} 0 & 0 \\ 0 & I_{\gamma_1} \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\gamma_2} \\ 0 & 0 & I_{\gamma_2} & 0 \\ 0 & I_{\gamma_2} & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{\gamma_3} \\ 0 & 0 & 0 & 0 & I_{\gamma_3} & 0 \\ 0 & 0 & 0 & I_{\gamma_3} & 0 & 0 \\ 0 & 0 & I_{\gamma_3} & 0 & 0 & 0 \\ 0 & I_{\gamma_3} & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\begin{aligned} & \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \mathcal{G}_3 = \hat{\mathcal{G}}_1 \oplus \hat{\mathcal{G}}_2 \oplus \hat{\mathcal{G}}_3 \\ = & \begin{bmatrix} 0 & I_{\gamma_1} \\ I_{\gamma_1} & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & I_{\gamma_2} \\ 0 & 0 & I_{\gamma_2} & 0 \\ 0 & I_{\gamma_2} & 0 & 0 \\ I_{\gamma_2} & 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I_{\gamma_3} \\ 0 & 0 & 0 & 0 & I_{\gamma_3} & 0 \\ 0 & 0 & 0 & I_{\gamma_3} & 0 & 0 \\ 0 & 0 & I_{\gamma_3} & 0 & 0 & 0 \\ 0 & I_{\gamma_3} & 0 & 0 & 0 & 0 \\ I_{\gamma_3} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \end{aligned}$$

$$\mathcal{A}_{1,2} \oplus \mathcal{A}_{2,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I_{n_1} \\ 0 & I_{m_1} & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_2} \\ 0 & 0 & 0 & I_{m_2} & 0 \\ 0 & 0 & I_{n_2} & 0 & 0 \\ 0 & I_{m_2} & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{G}_{1,2} \oplus \mathcal{G}_{2,3} = \begin{bmatrix} 0 & 0 & I_{m_1} \\ 0 & I_{n_1} & 0 \\ I_{m_1} & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 0 & I_{m_2} \\ 0 & 0 & 0 & I_{n_2} & 0 \\ 0 & 0 & I_{m_2} & 0 & 0 \\ 0 & I_{n_2} & 0 & 0 & 0 \\ I_{m_2} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{\mathcal{G}}_{1,2} \oplus \hat{\mathcal{G}}_{2,3} = \begin{bmatrix} 0 & 0 & I_{n_1} \\ 0 & I_{m_1} & 0 \\ I_{n_1} & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 0 & I_{n_2} \\ 0 & 0 & 0 & I_{m_2} & 0 \\ 0 & 0 & I_{n_2} & 0 & 0 \\ 0 & I_{m_2} & 0 & 0 & 0 \\ I_{n_2} & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 3) Extraction of Jordan blocks from the staircase-like-form

Completely analogous to the case $\ell = 3$, we proceed in the case $\ell \neq 3$ and obtain the staircase-like-form as

$$\begin{aligned} \widetilde{X}^T A \widetilde{Y} &= \mathcal{A}_{ns} \oplus \mathcal{A}_0 \oplus \bigoplus_{j=1}^{\ell} \mathcal{A}_j \oplus \bigoplus_{j=1}^{\ell-1} \mathcal{A}_{j,j+1}, \\ \widetilde{X}^T \widetilde{X} &= \mathcal{G}_{ns} \oplus \mathcal{G}_0 \oplus \bigoplus_{j=1}^{\ell} \mathcal{G}_j \oplus \bigoplus_{j=1}^{\ell-1} \mathcal{G}_{j,j+1}, \\ \widetilde{Y}^T \widetilde{Y} &= \hat{\mathcal{G}}_{ns} \oplus \hat{\mathcal{G}}_0 \oplus \bigoplus_{j=1}^{\ell} \hat{\mathcal{G}}_j \oplus \bigoplus_{j=1}^{\ell-1} \hat{\mathcal{G}}_{j,j+1}, \end{aligned}$$

where $\mathcal{A}_{ns}, \mathcal{G}_{ns}, \hat{\mathcal{G}}_{ns}$ are as in (7.11), $\mathcal{A}_0, \mathcal{G}_0, \hat{\mathcal{G}}_0$ are as in (7.12),

$$\mathcal{A}_j = \Big(R_{2j}\mathcal{J}_{2j}(0)\Big) \otimes I_{\gamma_j} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\gamma_j} \\ 0 & 0 & \cdot^{\cdot^{\cdot}} & 0 \\ 0 & I_{\gamma_j} & 0 & 0 \end{bmatrix}_{(2j)\times(2j)\text{ blocks}}, \tag{7.13}$$

$$\mathcal{G}_j = \hat{\mathcal{G}}_j = R_{2j} \otimes I_{\gamma_j} = \begin{bmatrix} 0 & 0 & I_{\gamma_j} \\ 0 & \cdot^{\cdot^{\cdot}} & 0 \\ I_{\gamma_j} & 0 & 0 \end{bmatrix}_{(2j)\times(2j)\text{ blocks}}, \tag{7.14}$$

and $\mathcal{A}_{j,j+1}$, $\hat{\mathcal{G}}_{j,j+1}$, and $\hat{\mathcal{G}}_{j,j+1}$ are $(2j+1) \times (2j+1)$ block matrices, where the block rows have alternating sizes n_j, m_j and the forms

$$\mathcal{A}_{j,j+1} = \begin{bmatrix} 0 & & & & & 0 \\ & & & & 0 & I_{n_j} \\ & & & \cdot^{\cdot^{\cdot}} & I_{m_j} & \\ & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & \\ & 0 & I_{n_j} & & & \\ 0 & I_{m_j} & & & & 0 \end{bmatrix}, \quad \mathcal{G}_{j,j+1} = \begin{bmatrix} 0 & & & & I_{m_j} \\ & & & I_{n_j} & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & & I_{\pi_j} & & \\ & \cdot^{\cdot^{\cdot}} & & & \\ & I_{n_j} & & & \\ I_{m_j} & & & & 0 \end{bmatrix}, \tag{7.15}$$

$$\hat{\mathcal{G}}_{j,j+1} = \begin{bmatrix} 0 & & & & I_{n_j} \\ & & & I_{m_j} & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & & I_{\hat{\pi}_j} & & \\ & \cdot^{\cdot^{\cdot}} & & & \\ & I_{m_j} & & & \\ I_{n_j} & & & & 0 \end{bmatrix}, \tag{7.16}$$

(Indeed, recall that $\pi_j, \hat{\pi}_j \in \{m_j, n_j\}$, where the actual definition depends on j being odd or even.) The blocks $\mathcal{A}_0$, $\mathcal{G}_0$, and $\hat{\mathcal{G}}_0$ are already in the form as indicated in Theorem 4.2 and we can apply Theorem 4.1 to the blocks $\mathcal{A}_{ns}$, $\mathcal{G}_{ns}$, and $\hat{\mathcal{G}}_{ns}$. Next, let us investigate in detail the blocks of the form (7.13)–(7.14). Let P_j be the permutation such that premultiplication with P_j^T reorders the rows of $\mathcal{A}_j$ in the order

$$\begin{array}{cccc} 2j\gamma_j, & (2j-1)\gamma_j, & \dots, & \gamma_j, \\ 2j\gamma_j - 1, & (2j-1)\gamma_j - 1, & \dots, & \gamma_j - 1, \\ \vdots & \vdots & \ddots & \vdots \\ 2j\gamma_j - \gamma_j + 1, & (2j-1)\gamma_j - \gamma_j + 1, & \dots, & 1; \end{array}$$

and let $\widetilde{P}_j$ be the permutation such that postmultiplication with $\widetilde{P}_j$ reorders the columns of $\mathcal{A}_j$ in the order

$$\begin{array}{cccc} \gamma_j, & \dots, & (2j-1)\gamma_j, & 2j\gamma_j, \\ \gamma_j - 1, & \dots, & (2j-1)\gamma_j - 1, & 2j\gamma_j - 1, \\ \vdots & \cdot^{\cdot^{\cdot}} & \vdots & \vdots \\ 1, & \dots, & (2j-1)\gamma_j - \gamma_j + 1, & 2j\gamma_j - \gamma_j + 1. \end{array}$$

Then it is easily verified that

$$P_j^T \mathcal{A}_j \widetilde{P}_j = \bigoplus_{i=1}^{\gamma_j} \mathcal{J}_{2j}(0), \quad P_j^T \mathcal{G}_j P_j = \widetilde{P}_j^T \hat{\mathcal{G}}_j \widetilde{P}_j = \bigoplus_{i=1}^{\gamma_j} R_{2j},$$

which is exactly the form of the blocks of type 2 in Theorem 4.2.

Finally, let us return to the blocks of the forms (7.15)–(7.16). Let Z_j be the permutation such that premultiplication with Z_j^T reorders the rows of $\mathcal{A}_{j,j+1}$ in the order

$$\begin{array}{ccccc}
(j+1)m_j + jn_j, & jm_j + (j-1)n_j, & \dots, & 2m_j + n_j, & m_j, \\
(j+1)m_j - 1 + jn_j, & jm_j - 1 + (j-1)n_j, & \dots, & 2m_j - 1 + n_j, & m_j - 1, \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
jm_j + 1 + jn_j, & (j-1)m_j + 1 + (j-1)n_j, & \dots, & m_j + 1 + n_j, & 1, \\
jm_j + jn_j, & (j-1)m_j + (j-1)n_j, & \dots, & m_j + n_j, & \\
jm_j + jn_j - 1, & (j-1)m_j + (j-1)n_j - 1, & \dots, & m_j + n_j - 1, & \\
\vdots & \vdots & \ddots & \vdots & \\
jm_j + (j-1)n_j + 1, & (j-1)m_j + (j-2)n_j + 1, & \dots, & m_j + 1, &
\end{array}$$

and let $\widetilde{Z}_{j+1}$ be the permutation such that postmultiplication with $\widetilde{Z}_{j+1}$ reorders the columns of $\mathcal{A}_{j,j+1}$ in the order

$$\begin{array}{ccccc}
m_j + n_j, & 2m_j + n_j, & \dots, & jm_j + jn_j, & \\
m_j - 1 + n_j, & 2m_j - 1 + n_j, & \dots, & jm_j - 1 + jn_j, & \\
\vdots & \vdots & \ddots & \vdots & \\
1 + n_j, & m_j + 1 + n_j, & \dots, & (j-1)m_j + 1 + jn_j, & \\
n_j, & m_j + 2n_j, & \dots, & (j-1)m_j + jn_j, & jm_j + (j+1)n_j, \\
n_j - 1, & m_j + 2n_j - 1, & \dots, & (j-1)m_j + jn_j - 1, & jm_j + (j+1)n_j - 1, \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1, & m_j + n_j + 1, & \dots, & (j-1)m_j + (j-1)n_j + 1, & jm_j + jn_j + 1.
\end{array}$$

Then it is easily verified that

$$\begin{aligned}
Z_j^T \mathcal{A}_{j,j+1} \widetilde{Z}_{j+1} &= \bigoplus_{i=1}^{m_j} \begin{bmatrix} I_j \\ 0 \end{bmatrix}_{(j+1)\times j} \oplus \bigoplus_{i=1}^{n_j} \begin{bmatrix} 0 & I_j \end{bmatrix}_{j\times(j+1)}, \\
Z_j^T \mathcal{G}_{j,j+1} Z_j &= \bigoplus_{i=1}^{m_j} R_{j+1} \oplus \bigoplus_{i=1}^{n_j} R_j, \\
\widetilde{Z}_{j+1}^T \hat{\mathcal{G}}_{j,j+1} \widetilde{Z}_{j+1} &= \bigoplus_{i=1}^{m_j} R_j \oplus \bigoplus_{i=1}^{n_j} R_{j+1},
\end{aligned} \tag{7.17}$$

and we have obtained the blocks as in 3) and 4) of Theorem 4.2.

Up to this point, we have proved the existence of the canonical form for the triple $(A, G, \hat{G})$. The corresponding forms for $\hat{\mathcal{H}}$ and $\mathcal{H}$ then immediately from forming the products $\hat{G}^{-1}A^T G^{-1} A$ and $G^{-1} A \hat{G}^{-1} A^T$.

Step 4) Uniqueness of the form

Concerning uniqueness, in view of Theorem 4.1 it remains to show that the numbers ℓ_j, m_j, n_j are uniquely determined. Note that there exists a unique sequence of subspaces

$$\mathrm{Eig}_{\nu}(\hat{\mathcal{H}}, 0) \subseteq \mathrm{Eig}_{\nu-1}(\hat{\mathcal{H}}, 0) \subseteq \cdots \subseteq \mathrm{Eig}_{1}(\hat{\mathcal{H}}, 0) = \ker \hat{\mathcal{H}}$$

where $\mathrm{Eig}_j(\hat{\mathcal{H}}, 0)$ consists the zero vector and of all eigenvectors of $\hat{\mathcal{H}}$ associated with zero that can be extended to a Jordan chain of length at least j. Define $\kappa_\nu = \dim\big(\mathrm{Eig}_\nu(\hat{\mathcal{H}}, 0) \cap \ker A\big)$ and

$$\kappa_j = \dim\big(\mathrm{Eig}_j(\hat{\mathcal{H}}, 0) \cap \ker A\big) - \dim\big(\mathrm{Eig}_{j+1}(\hat{\mathcal{H}}, 0) \cap \ker A\big), \quad j = 1, \ldots, \nu - 1.$$

Then any eigenvector of $\hat{\mathcal{H}}$ that is associated with a Jordan block of size $j \times j$ in the canonical form and that is also in the kernel of A contributes to κ_j. Similarly, we define $\hat{\kappa}_\nu = \dim\big(\mathrm{Eig}_\nu(\mathcal{H}, 0) \cap \ker A^T\big)$ and

$$\hat{\kappa}_j = \dim\big(\mathrm{Eig}_j(\mathcal{H}, 0) \cap \ker A^T\big) - \dim\big(\mathrm{Eig}_{j+1}(\mathcal{H}, 0) \cap \ker A^T\big), \quad j = 1, \ldots, \nu - 1.$$

Then elementary counting yields

$$\kappa_j = \ell_j + n_{j-1} \quad \text{and} \quad \hat{\kappa}_j = \ell_j + m_{j-1}, \quad j = 1, \ldots, \nu.$$

If p_j respectively $\hat{p}_j$ denote the number of Jordan blocks of size $j \times j$ in the canonical form of $\hat{\mathcal{H}}$ and $\mathcal{H}$, respectively, we also have that

$$p_j = 2\ell_j + m_j + n_{j-1} \quad \text{and} \quad \hat{p}_j = 2\ell_j + m_{j-1} + n_j, \quad j = 1, \ldots, \nu.$$

Hence, we obtain

$$p_j - \kappa_j - \hat{\kappa}_j = m_j - m_{j-1}, \quad \text{and} \quad \hat{p}_j - \kappa_j - \hat{\kappa}_j = n_j - n_{j-1}, \quad j = 1, \ldots, \nu,$$

from which we can successively compute m_j, n_j, $j = \nu - 1, \ldots, 0$ using $m_\nu = n_\nu = 0$. We furthermore obtain that

$$\ell_j = \frac{1}{2}(p_j - m_j - n_{j-1})$$

for $j = 1, \ldots, \nu$. Thus, the numbers ℓ_j, m_j, n_j are uniquely determined by the invariant numbers $p_j, \hat{p}_j, \kappa_j, \hat{\kappa}_j$, $j = 1, \ldots, \nu$.

This concludes the proof of Theorem 4.2. □

Proof of Theorem 5.2

Applying appropriate congruence transformations to G and $\hat{G}$ otherwise, we may assume that $G = I_m$ and $\hat{G} = J_n$. Let

$$A = B_1 C_1^T$$

be a full rank factorization of A, i.e., $B_1 \in \mathbb{C}^{m \times r}$, $C_1 \in \mathbb{C}^{2n \times r}$, $\operatorname{rank} B_1 = \operatorname{rank} C_1 = r$. Repeatedly applying Proposition 3.3 to B_1 and Proposition 3.7 to C_1, respectively, we can determine a staircase-like form that can be further reduced to canonical form. The proof follows the same lines as in the steps 1) and 2) of the proof of Theorem 4.2 and yields the reduced staircase-like form

$$\begin{aligned}
\widetilde{X}^T A \widetilde{Y} &= \mathcal{A}_{ns} \oplus \mathcal{A}_0 \oplus \bigoplus_{j=1}^{\ell} \mathcal{A}_j \oplus \bigoplus_{j=1}^{\ell-1} \mathcal{A}_{j,j+1}, \\
\widetilde{X}^T \widetilde{X} &= \mathcal{G}_{ns} \oplus \mathcal{G}_0 \oplus \bigoplus_{j=1}^{\ell} \mathcal{G}_j \oplus \bigoplus_{j=1}^{\ell-1} \mathcal{G}_{j,j+1}, \\
\widetilde{Y}^T J_n \widetilde{Y} &= \hat{\mathcal{G}}_{ns} \oplus \hat{\mathcal{G}}_0 \oplus \bigoplus_{j=1}^{\ell} \hat{\mathcal{G}}_j \oplus \bigoplus_{j=1}^{\ell-1} \hat{\mathcal{G}}_{j,j+1},
\end{aligned}$$

where

$$\mathcal{A}_{ns} = A_{2\ell+1,2\ell+1}, \quad \mathcal{G}_{ns} = I_{\pi_\ell} = I_{2\hat{\pi}_\ell}, \quad \hat{\mathcal{G}}_{ns} = J_{\hat{\pi}_\ell},$$

with $A_{2\ell+1,2\ell+1} \in \mathbb{C}^{\pi_\ell \times \pi_\ell}$ being nonsingular,

$$\mathcal{A}_0 = 0_{m_0 \times 2n_0}, \quad \mathcal{G}_0 = I_{m_0}, \quad \hat{\mathcal{G}}_0 = J_{n_0},$$

$$\mathcal{A}_j = \Big(R_{2j} \mathcal{J}_{2j}(0)\Big) \otimes I_{\gamma_j}, \quad \mathcal{G}_j = R_{2j} \otimes I_{\gamma_j} \quad , \hat{\mathcal{G}}_j = \begin{bmatrix} 0 & R_j \\ -R_j & 0 \end{bmatrix} \otimes I_{\gamma_j},$$

and $\mathcal{A}_{j,j+1}$, $\hat{\mathcal{G}}_{j,j+1}$, and $\hat{\mathcal{G}}_{j,j+1}$ are $(2j+1) \times (2j+1)$ block matrices, where, if j is odd, the block rows have alternating sizes $n_j, 2m_j$ and the forms

$$\mathcal{A}_{j,j+1} = \begin{bmatrix} 0 & & & & & 0 \\ & & & & 0 & I_{n_j} \\ & & & \iddots & I_{2m_j} & \\ & & \iddots & \iddots & & \\ & 0 & I_{n_j} & & & \\ 0 & I_{2m_j} & & & & 0 \end{bmatrix}, \quad \mathcal{G}_{j,j+1} = \begin{bmatrix} 0 & & & & & I_{2m_j} \\ & & & & I_{n_j} & \\ & & & \iddots & & \\ & & I_{n_j} & & & \\ & \iddots & & & & \\ I_{n_j} & & & & & \\ I_{2m_j} & & & & & 0 \end{bmatrix}, \quad (7.18)$$

$$\hat{\mathcal{G}}_{j,j+1} = \begin{bmatrix} 0 & & & & & I_{n_j} \\ & & & & I_{2m_j} & \\ & & & \iddots & & \\ & & J_{m_j} & & & \\ & \iddots & & & & \\ & -I_{2m_j} & & & & \\ -I_{n_j} & & & & & 0 \end{bmatrix}, \quad (7.19)$$

or, if j is even, then the block rows have alternating sizes $2n_j, m_j$ and the forms

$$
\mathcal{A}_{j,j+1} = \begin{bmatrix} 0 & & & & & 0 \\ & & & & 0 & I_{2n_j} \\ & & & \iddots & I_{m_j} & \\ & & \iddots & \iddots & & \\ & 0 & I_{2n_j} & & & \\ 0 & I_{m_j} & & & & 0 \end{bmatrix}, \quad \mathcal{G}_{j,j+1} = \begin{bmatrix} 0 & & & & I_{m_j} \\ & & & I_{2n_j} & \\ & & \iddots & & \\ & I_{m_j} & & & \\ I_{2n_j} & & & & \\ I_{m_j} & & & & 0 \end{bmatrix}, \tag{7.20}
$$

$$
\hat{\mathcal{G}}_{j,j+1} = \begin{bmatrix} 0 & & & & I_{2n_j} \\ & & & I_{m_j} & \\ & & \iddots & & \\ & & J_{n_j} & & \\ & \iddots & & & \\ & -I_{m_j} & & & \\ -I_{2n_j} & & & & 0 \end{bmatrix}, \tag{7.21}
$$

The blocks $\mathcal{A}_0$, $\mathcal{G}_0$, and $\hat{\mathcal{G}}_0$ are already in the form as indicated in Theorem 5.2, for the blocks $\mathcal{A}_{ns}$, $\mathcal{G}_{ns}$, $\hat{\mathcal{G}}_{ns}$, we can apply Theorem 5.1, and for the blocks $\mathcal{A}_j$, $\hat{\mathcal{G}}_j$, and $\hat{\mathcal{G}}_j$ we can apply an analogous permutation as it has been done for the corresponding blocks in the proof of Theorem 4.2. Moreover, if j is odd, then let Z_j be the permutation such that premultiplication with Z_j^T reorders the rows of $\mathcal{A}_{j,j+1}$ in the order

$$
\begin{array}{ccccc}
2(j+1)m_j + jn_j, & 2jm_j + (j-1)n_j, & \ldots, & 4m_j + n_j, & 2m_j, \\
2jm_j + m_j + jn_j, & 2(j-1)m_j + m_j + (j-1)n_j, & \ldots, & 2m_j + m_j + n_j, & m_j, \\
2(j+1)m_j - 1 + jn_j, & 2jm_j - 1 + (j-1)n_j, & \ldots, & 4m_j - 1 + n_j, & 2m_j - 1, \\
2jm_j + m_j - 1 + jn_j, & 2(j-1)m_j + m_j - 1 + (j-1)n_j, & \ldots, & 2m_j + m_j - 1 + n_j, & m_j - 1, \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
2jm_j + m_j + 1 + jn_j, & 2(j-1)m_j + m_j + 1 + (j-1)n_j, & \ldots, & 2m_j + m_j + 1 + n_j, & m_j + 1, \\
2jm_j + 1 + jn_j, & 2(j-1)m_j + 1 + (j-1)n_j, & \ldots, & 2m_j + 1 + n_j, & 1, \\
2jm_j + jn_j, & 2(j-1)m_j + (j-1)n_j, & \ldots, & 2m_j + n_j, & \\
2jm_j + jn_j - 1, & 2(j-1)m_j + (j-1)n_j - 1, & \ldots, & 2m_j + n_j - 1, & \\
\vdots & \vdots & \ddots & \vdots & \\
2jm_j + (j-1)n_j + 1, & 2(j-1)m_j + (j-2)n_j + 1, & \ldots, & 2m_j + 1, &
\end{array}
$$

and let $\widetilde{Z}_{j+1}$ be the permutation such that postmultiplication with $\widetilde{Z}_{j+1}$ reorders the columns of $\mathcal{A}_{j,j+1}$ in the order

$$
\begin{array}{ccccc}
m_j + n_j, & 2m_j + m_j + 2n_j, & \ldots, & 2(j-1)m_j + m_j + jn_j, & \\
2m_j + n_j, & 4m_j + n_j, & \ldots, & 2jm_j + jn_j, & \\
m_j - 1 + n_j, & 2m_j + m_j - 1 + 2n_j, & \ldots, & 2(j-1)m_j + m_j - 1 + jn_j, & \\
2m_j - 1 + n_j, & 4m_j - 1 + n_j, & \ldots, & 2jm_j - 1 + jn_j, & \\
\vdots & \vdots & \ddots & \vdots & \\
1 + n_j, & 2m_j + 1 + n_j, & \ldots, & 2(j-1)m_j + 1 + jn_j, & \\
m_j + 1 + n_j, & 2m_j + m_j + 1 + n_j, & \ldots, & 2(j-1)m_j + m_j + 1 + jn_j, & \\
n_j, & 2m_j + 2n_j, & \ldots, & 2(j-1)m_j + jn_j, & 2jm_j + (j+1)n_j, \\
n_j - 1, & 2m_j + 2n_j - 1, & \ldots, & 2(j-1)m_j + jn_j - 1, & 2jm_j + (j+1)n_j - 1, \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1, & 2m_j + n_j + 1, & \ldots, & 2(j-1)m_j + (j-1)n_j + 1, & 2jm_j + jn_j + 1.
\end{array}
$$

(Notice the slight difference in the way how the permutation matrices Z_j and $\widetilde{Z}_j$ are build up compared to the way in the proof of Theorem 4.2. In this way, we can group together two paired blocks of equal size into one block.) Then it is easily verified that

$$\begin{aligned}
Z_j^T \mathcal{A}_{j,j+1} \widetilde{Z}_{j+1} &= \bigoplus_{i=1}^{m_j} \begin{bmatrix} 0 & I_j \\ 0 & 0 \\ I_j & 0 \\ 0 & 0 \end{bmatrix}_{2(j+1)\times 2j} \oplus \bigoplus_{i=1}^{n_j} \begin{bmatrix} 0 & I_j \end{bmatrix}_{j\times(j+1)}, \\
Z_j^T \mathcal{G}_{j,j+1} Z_j &= \bigoplus_{i=1}^{m_j} \begin{bmatrix} 0 & R_{j+1} \\ R_{j+1} & 0 \end{bmatrix} \oplus \bigoplus_{i=1}^{n_j} R_j, \\
\widetilde{Z}_{j+1}^T \hat{\mathcal{G}}_{j,j+1} \widetilde{Z}_{j+1} &= \bigoplus_{i=1}^{m_j} \begin{bmatrix} 0 & R_j \\ -R_j & 0 \end{bmatrix} \oplus \bigoplus_{i=1}^{n_j} \begin{bmatrix} 0 & R_{\frac{j+1}{2}} \\ -R_{\frac{j+1}{2}} & 0 \end{bmatrix}
\end{aligned} \tag{7.22}$$

i.e., we obtain blocks as in 4) and 5) in Theorem 5.2. Similarly, an analogous permutation extracts blocks as in 3) and 6) in Theorem 5.2 for the case that j is even, i.e., if we consider the blocks (7.20)–(7.21). (In the theorem, for cosmetic reasons we changed the meaning of ℓ by letting ℓ be such that $2\ell+1$ is the smallest odd number that is larger than or equal to the maximum of the indices of $\hat{\mathcal{H}}$ and $\mathcal{H}$.)

Concerning uniqueness, as in the proof of Theorem 4.2 it remains to show uniqueness of the numbers ℓ_j, $2m_j$, and n_j. This is done exactly in the same way as in the proof of Theorem 4.2. Note that the paired blocks in 4) and 6) in Theorem 5.2 cannot be decomposed into two smaller blocks of equal size, because of the fact that nonsingular skew-symmetric matrices must have even size. □

Proof of Theorem 6.2

Applying appropriate congruence transformations to G and $\hat{G}$ otherwise, we may assume that $G = J_m$ and $\hat{G} = J_n$. Again, we then compute a staircase-like form for A by considering the full rank factorization

$$A = B_1 C_1^T$$

of A, i.e., $B_1 \in \mathbb{C}^{2m\times r}$, $C_1 \in \mathbb{C}^{2n\times r}$, $\operatorname{rank} B_1 = \operatorname{rank} C_1 = r$, and repeatedly applying Proposition 3.7 to B_1 and C_1. Then continuing as in step 2) of the proof of Theorem 4.2 yields the reduced staircase-like form

$$\begin{aligned}
\widetilde{X}^T A \widetilde{Y} &= \mathcal{A}_{ns} \oplus \mathcal{A}_0 \oplus \bigoplus_{j=1}^{\ell} \mathcal{A}_j \oplus \bigoplus_{j=1}^{\ell-1} \mathcal{A}_{j,j+1}, \\
\widetilde{X}^T J_m \widetilde{X} &= \mathcal{G}_{ns} \oplus \mathcal{G}_0 \oplus \bigoplus_{j=1}^{\ell} \mathcal{G}_j \oplus \bigoplus_{j=1}^{\ell-1} \mathcal{G}_{j,j+1}, \\
\widetilde{Y}^T J_n \widetilde{Y} &= \hat{\mathcal{G}}_{ns} \oplus \hat{\mathcal{G}}_0 \oplus \bigoplus_{j=1}^{\ell} \hat{\mathcal{G}}_j \oplus \bigoplus_{j=1}^{\ell-1} \hat{\mathcal{G}}_{j,j+1},
\end{aligned}$$

where

$$\mathcal{A}_{ns} = A_{2\ell+1,2\ell+1}, \quad \mathcal{G}_{ns} = J_{\pi_\ell}, \quad \hat{\mathcal{G}}_{ns} = J_{\hat{\pi}_\ell} = J_{\pi_\ell},$$

with $A_{2\ell+1,2\ell+1} \in \mathbb{C}^{2\pi_\ell \times 2\pi_\ell}$ being nonsingular,

$$\mathcal{A}_0 = 0_{2m_0 \times 2n_0}, \quad \mathcal{G}_0 = J_{m_0}, \quad \hat{\mathcal{G}}_0 = J_{n_0},$$

$$\mathcal{A}_j = \Big(R_{2j} \mathcal{J}_{2j}(0) \Big) \otimes I_{\gamma_j}, \quad \mathcal{G}_j = \hat{\mathcal{G}}_j = \begin{bmatrix} 0 & R_j \\ -R_j & 0 \end{bmatrix} \otimes I_{\gamma_j},$$

and $\mathcal{A}_{j,j+1}$, $\hat{\mathcal{G}}_{j,j+1}$, and $\hat{\mathcal{G}}_{j,j+1}$ are $(2j+1) \times (2j+1)$ block matrices, where the block rows have alternating sizes $2n_j, 2m_j$ and the forms

$$\mathcal{A}_{j,j+1} = \begin{bmatrix} 0 & & & & & 0 \\ & & & & 0 & I_{2n_j} \\ & & & \iddots & I_{2m_j} & \\ & & \iddots & \iddots & & \\ & 0 & I_{2n_j} & & & \\ 0 & I_{2m_j} & & & & 0 \end{bmatrix}, \quad \mathcal{G}_{j,j+1} = \begin{bmatrix} 0 & & & & I_{2m_j} \\ & & & I_{2n_j} & \\ & & J_{\pi_j} & & \\ & -I_{2m_j} & & & \\ -I_{2n_j} & & & & 0 \end{bmatrix}, \tag{7.23}$$

$$\hat{\mathcal{G}}_{j,j+1} = \begin{bmatrix} 0 & & & & I_{2n_j} \\ & & & I_{2m_j} & \\ & & J_{\hat{\pi}_j} & & \\ & -I_{2m_j} & & & \\ -I_{2n_j} & & & & 0 \end{bmatrix}, \tag{7.24}$$

The remainder of the proof then proceed as the proof of Theorem 4.2 by adapting the permutation used on the blocks of the forms (7.23)–(7.24) similarly as in the proof of Theorem 6.2 in order to allow to group together paired blocks.

Concerning uniqueness, as in the proof of Theorem 4.2 it remains to show uniqueness of the numbers ℓ_j, $2m_j$, and $2n_j$. This is done exactly in the same way as in the proof of Theorem 4.2. $\square$