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Numerical Methods in Control

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Abstract

We study classical control problems like pole assignment, stabilization, linear quadratic control and H_∞ control from a numerical analysis point of view. We present several examples that show the difficulties with classical approaches and suggest reformulations of the problems in a more general framework. We also discuss some new algorithmic approaches.

Keywords pole placement, linear quadratic control, stabilization H_∞ control, algebraic Riccati equation, Hamiltonian matrix, skew Hamiltonian matrix, two point boundary value problem
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1 Introduction

In the last 40 years systems and control theory has evolved into a mature field that has found a stable position on the borderline between applied mathematics, engineering and computer science. The major success is not only due to the fact that beautiful mathematical theories (like linear algebra, ring theory, representation theory and others) find direct application but also since the results have immediately found their ways into production code software packages like MATLAB toolboxes [55, 56] or the SLICOT subroutine library [13], which can be and are directly used by engineers working

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in practice. In this paper we will discuss several problems of linear control theory, as there are *pole assignment, stabilization, linear quadratic control and H_∞ control.* In the solution techniques for these problems important developments have taken place in recent years, which have lead to changes in viewpoints in particular what the numerical solution of these problems is concerned. In our opinion there are three central questions that need to be studied in more detail in the context of numerical methods for the solution of control problems and it is the aim of this paper to initiate more research and software developments in this direction.

First of all, as is well-known, different mathematically equivalent formulations of the same problem may lead to drastically different sensitivity of the problem to perturbations (such as round-off errors) and thus it is important to find the best formulation for numerical solution.

The second issue is that the numerical methods should reflect the physical properties of the problem in the maximal way, to get higher efficiency but also to guarantee even in finite arithmetic that the computed results are physically meaningful.

The third important topic is that with the growing complexity of problems, in particular in the context of large scale control problems, solution approaches and numerical methods have to be reviewed and completely new methods have to be developed.

We will only discuss the first two issues but large scale control problems are currently a very important research topic.

Consider linear constant coefficient dynamical systems of the form

$$\dot{x} = Ax + Bu, \qquad x(t_0) = x^0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, x^0 is an initial vector, $u(t) \in \mathbb{R}^m$ is the control input of the system and the matrices $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$ are constant. The topics that we discuss here also apply in a similar fashion to problems with output and also to complex problems, but for the sake of brevity we only discuss real problems.

The classical **pole placement problem** is to find a state feedback control law

$$u = Kx \tag{2}$$

such that the closed loop system

$$\dot{x} = (A + BK)x \tag{3}$$

has desired poles, or in linear algebra terminology, that the spectrum of the closed loop system matrix $A + BF$ is a given set of complex numbers. Here,

the case of **stabilization**, where the closed loop poles are desired to be in the open left half plane represents an important special case.

For a discussion of the classical theory of the pole placement problem and related problems, we refer the reader to monographs in linear control theory, e.g., [7, 28, 42, 45, 51, 66, 86]. In Section 2 we discuss some new perturbation results and the resulting consequences for numerical methods. These results indicate that the numerical solution of the classical formulation of the pole placement problem is often and in particular for large n and small m a highly ill-conditioned problem that should be modified.

This analysis and the resulting conclusions hold also for the stabilization problem which alternatively may be solved also via the solution of a **linear quadratic control problem**. For this the objective is to find a control law $u(t)$ such that the closed loop system is asymptotically stable and such that the performance criterion

$$\mathcal{S}(x,u) = \int_{t_0}^{\infty} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & L \\ L^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \tag{4}$$

is minimized, where $Q = Q^T \in \mathbb{R}^{n,n}$, $R = R^T \in \mathbb{R}^{m,m}$ is positive definite and $\begin{bmatrix} Q & L \\ L^T & R \end{bmatrix}$ is positive semidefinite.

The basics for this problem can be found in classical monographs on linear control [4, 7, 16, 28, 52, 42, 45, 51, 59, 66, 74, 86].

Application of the maximum principle [59, 70] leads to the problem of finding a stable solution to the two-point boundary value problem of Euler-Lagrange equations

$$\mathcal{E}_c \begin{bmatrix} \dot{x} \\ \dot{\mu} \\ \dot{u} \end{bmatrix} = \mathcal{A}_c \begin{bmatrix} x \\ \mu \\ u \end{bmatrix}, \qquad x(t_0) = x^0, \qquad \lim_{t\to\infty} \mu(t) = 0, \tag{5}$$

with the matrix pencil

$$\alpha\mathcal{E}_c - \beta\mathcal{A}_c := \alpha \begin{bmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} A & 0 & B \\ Q & A^T & L \\ L^T & B^T & R \end{bmatrix}. \tag{6}$$

If R is well-conditioned with respect to inversion, then (5) may be reduced to the two-point boundary value problem

$$\begin{bmatrix} \dot{x} \\ -\dot{\mu} \end{bmatrix} = \mathcal{H} \begin{bmatrix} x \\ -\mu \end{bmatrix}, \qquad x(t_0) = x^0, \qquad \lim_{t\to\infty} \mu(t) = 0 \tag{7}$$

with the *Hamiltonian matrix*

$$\mathcal{H} = \begin{bmatrix} F & G \\ H & -F^T \end{bmatrix} := \begin{bmatrix} A - BR^{-1}L^T & BR^{-1}B^T \\ Q - LR^{-1}L^T & -(A - BR^{-1}L^T)^T \end{bmatrix}. \quad (8)$$

The solution of the boundary value problems (5) and (7) can be obtained in many different ways. The classical way, that is implemented in most design packages is to determine first X, the positive semidefinite (stabilizing) solution of the associated algebraic Riccati equation

$$0 = H + XF + F^T X - XGX, \quad (9)$$

and then obtaining the optimal stabilizing feedback as

$$u(t) = -R^{-1}B^T X x(t). \quad (10)$$

The solution of the algebraic Riccati equation is also often used for the decoupling of the forward and backward integration. But one may also directly solve the two point boundary value problem (5) or alternatively (7) without going via the Riccati equation and we will show in Section 3 that this is actually numerically a much better approach and that the Riccati equation presents an unnecessary and sometimes dangereous detour.

As we have already mentioned, we may use both linear quadratic control and pole placement for the objective of stabilization. In Section 4, we compare pole assignment and the solution of linear quadratic control problems for stabilization.

The third problem that we include into our discussion is the standard H_∞ **control problem** which arises in the context of robust control in frequency domain, see, e.g., the recent monographs [34, 88]. In this problem one studies the linear system

$$\begin{aligned} \dot{x} &= Ax + B_1 u + B_2 w, \quad x(t_0) = x^0, \\ z &= C_1 x + D_{11} u + D_{12} w, \\ y &= C_2 x + D_{21} u + D_{22} w, \end{aligned} \quad (11)$$

where $A \in \mathbb{R}^{n,n}$, $B_k \in \mathbb{R}^{n,m_k}$, $C_k \in \mathbb{R}^{p_k,n}$ for $k = 1, 2$, and $D_{ij} \in \mathbb{R}^{p_i,m_j}$ for $i, j = 1, 2$. Here $w(t) \in \mathbb{R}^{m_2}$ describes noise, modelling errors or an unknown part of the system, $y(t) \in \mathbb{R}^{p_2}$ describes measured outputs while $z \in \mathbb{R}^{p_1}$ describes the regulated outputs. The objective of optimal H_∞ control is to find a control law

$$\begin{aligned} \dot{q} &= \hat{A}q + \hat{B}y \\ u &= \hat{C}q + \hat{D}y \end{aligned} \quad (12)$$

to minimize the closed loop transfer function T_{zw} from w to z in H_∞ norm.

Under some technical assumptions, see [88] or [31] for the general case, for a given parameter $\gamma > 0$, a necessary and sufficient condition for the existence of an admissible controller such that $\|T_{zw}\|_\infty < \gamma$, is that the following conditions hold (e.g., [88, Theorem 16.4, p. 419]):

(A1) For the matrix

$$\mathcal{H}_\infty := \begin{bmatrix} A & \gamma^{-2}B_1B_1^T - B_2B_2^T \\ -C_1^TC_1 & -A^T \end{bmatrix}, \tag{13}$$

there exists matrices $Q_1, Q_2 \in \mathbb{R}^{n,n}$ such that

$$\mathcal{H}_\infty \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} T_x, \tag{14}$$

where T_x has only eigenvalues with non positive real parts, Q_1 is nonsingular, and $X_\infty := Q_2Q_1^{-1}$ is symmetric positive semidefinite.

(A2) For the matrix

$$\mathcal{J}_\infty := \begin{bmatrix} A & -B_1B_1^T \\ \gamma^{-2}C_1^TC_1 - C_2^TC_2 & -A^T \end{bmatrix}, \tag{15}$$

there exist matrices $U_1, U_2 \in \mathbb{R}^{n,n}$ such that

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}^T \mathcal{J}_\infty = T_y \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}^T, \tag{16}$$

where T_y has only eigenvalues with non positive real parts, U_1 is nonsingular, and $Y_\infty := U_2U_1^{-1}$ is symmetric positive semidefinite.

(A3) For the matrices X_∞, Y_∞ we have that $\gamma^2 > \rho(X_\infty Y_\infty)$, where $\rho(A)$ denotes the spectral radius of the matrix A.

The optimal H_∞ control is then obtained by finding the smallest admissable γ so that conditions (A1)–(A3) still hold. The optimal controller yields system (12) with

$$\begin{aligned} \hat{A} &:= A + \gamma^{-2}B_1B_1^TX_\infty + B_2\hat{C} - \hat{B}C_2, \\ \hat{B} &:= (I - \gamma^{-2}Y_\infty X_\infty)^{-1}Y_\infty C_2^T, \quad \hat{C} := -B_2^TX_\infty, \quad \hat{D} := 0. \end{aligned} \tag{17}$$

We see that for the conditions (A1), (A2) we have Hamiltonian matrices which (except for the indefiniteness of blocks) are similar to the Hamiltonians

arising in the linear quadratic control problem, and hence the analysis and improvements for the linear quadratic control problem also hold for the H_∞ problem. We discuss this topic in Section 6.

Before going into details, let us recall that we have the following objectives in mind. We want to determine the best formulation of the problem for the use in numerical solution methods and furthermore we wish to obtain methods that are best adapted to all the underlying physical and mathematical structures in order to obtain efficient and accurate solution methods.

2 Pole Placement

As we have discussed in the introduction, in linear algebra terminology the pole placement problem is as follows:

Problem 1 *For given matrices $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$ and a given set of n complex numbers $\mathcal{P} = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$, that is closed under conjugation, find a matrix $K \in \mathbb{R}^{m,n}$, such that the set of eigenvalues of $A + BK$ is equal to $\mathcal{P}$.*

It is well-known, see e.g., [42, 85], that a *feedback gain* matrix K exists for all possible sets $\mathcal{P} \subset \mathbb{C}$, that are closed under conjugation if and only if (A, B) is *controllable*, i.e.,

$$\operatorname{rank}[A - \lambda I_n, B] = n, \ \forall \lambda \in \mathbb{C}. \tag{18}$$

There is a large literature on this problem. Extensions of Ackermann's explicit formula [1] for the single-input case were given in [61, 79] and also many numerical algorithms were developed for this problem, see [43, 64, 67, 73, 83]. For some of these methods, numerical backward stability has been established, see e.g. [6, 26, 27, 43, 64, 67]. However, it is nevertheless often observed that the numerical results are very inaccurate. If a numerically stable method yields highly inaccurate results then this is due to ill-conditioning of the problem. Therefore the conditioning of the pole placement problem was analyzed but the conclusions from the analysis are quite different, see [5, 36, 46, 48], and there are several reasons for these differences.

First of all pole assignment is usually approached via a two-step procedure, which first brings the pair (A, B) to a simpler form and then assigns the poles in this simpler form. But in such a two-step procedure it may sometimes happen that although the original problem was well-conditioned (i.e., small perturbations in the data only lead to small changes in the solution), one of the intermediate steps is very ill-conditioned. To avoid this

problem a good method for the initial reduction has to be used. For the pole assignment problem the best reduction is given by the staircase form of Van Dooren [80] or variations of it, see [47], which essentially does not affect the perturbations except for situations where the problem is very near to an uncontrollable problem, i.e., a problem (A, B) for which the *distance to uncontrollability* defined as

$$d_u(A, B) := \min_{\lambda \in \mathcal{C}} \sigma_n[A - \lambda I, B], \tag{19}$$

see [30], is small. Here $\sigma_n(A)$ is the smallest singular value of the matrix A. Since controllability is the necessary and sufficient condition for solvability of the pole placement problem, it is clear that a problem that is near to an uncontrollable problem will be very sensitive to perturbations. Hence the distance to uncontrollability (if small) is an important factor in the perturbation analysis of the pole placement problem but, as we will see below, other factors are equally or even more important.

The second reason for confusion in the evaluation of the pole placement problem is that one has to define clearly what the *solution* of the problem is. This could be the feedback K, the closed loop matrix $A+BK$ or its spectrum, respectively. All of these are solutions of the pole placement problem but they exhibit largely different pertubation results. A striking example of a stabilization problem is the case $m = 1$ in Example 1 below, see also [60], which shows that even though the feedback K is computed analytically, and the distance to uncontrollability is large, the (presumingly) stabilized closed loop system has eigenvalues with positive real part, something which could be a disaster in a practical application.

In our opinion the most important goal of pole placement is that the poles of the closed loop system obtained with the computed feedback are close to the desired ones and in the case of stabilization the resulting closed loop system is robustly stable. If the desired poles of the exact closed loop system are very sensitive to perturbations then this ultimate goal usually cannot be guaranteed. And this may happen even if the computation of K is reliable or even exact.

With this goal in mind, a new analysis and new explicit solution formulas that cover all the aspects of the problem have recently been given in [60, 61] and we will interpret some of these results here. The major conclusions can be obtained from the following result which generalizes a perturbation result of [77]. For this result we need the scaled spectral condition number of a matrix A given by $\|TD\|\|(TD)^{-1}\|$, where T is the matrix that transforms A to Jordan canonical form and D is a diagonal matrix that scales the columns

of T to have all unit norm, see [29].

Theorem 1 *[61] Consider a controllable matrix pair (A, B), and a set of poles $\mathcal{P} = \{\lambda_1, \ldots, \lambda_n\}$. Consider a perturbed system $(\hat{A}, \hat{B})$ which is also controllable and a perturbed set of poles $\hat{\mathcal{P}} = \{\hat{\lambda}_1, \ldots, \hat{\lambda}_n\}$. Set $\hat{A} - A =: \delta A$, $\hat{B} - B =: \delta B$ and $\hat{\lambda}_k - \lambda_k =: \delta\lambda_k$, $k = 1, \ldots, n$. Suppose that both the pole placement problems with $A, B, \mathcal{P}$ and $\hat{A}, \hat{B}, \hat{\mathcal{P}}$ have solutions with a diagonalizable closed loop matrix. Set*

$$\epsilon := \|[\delta A, \delta B]\| \tag{20}$$

and suppose that

$$\max_i \frac{\epsilon + |\delta\lambda_i|}{\sigma_n([A - \lambda_i I, B])} < \frac{3}{4}. \tag{21}$$

Then there exists a feedback gain $\hat{K} := K + \delta K$ of $(\hat{A}, \hat{B})$ such that

$$\|\delta K\| < \frac{5\sqrt{n}}{4}\kappa\sqrt{1 + \left\|\hat{K}\right\|^2} \max_i \left\{ \frac{\sqrt{1 + (\|B^\dagger(A - \lambda_i I)\|)^2}\,(\epsilon + |\delta\lambda_i|)}{\sigma_n([A - \lambda_i I, B])} \right\}, \tag{22}$$

the spectrum of $(\hat{A} + \hat{B}\hat{K})$ is $\hat{\mathcal{P}}$ and $\hat{A} + \hat{B}\hat{K}$ is diagonalizable.

Moreover, for each eigenvalue μ_i of the closed loop matrix $A + B\hat{K}$, (i.e., the perturbed feedback is used for the unperturbed system), there is a corresponding $\lambda_i \in \mathcal{P}$ such that

$$|\mu_i - \lambda_i| < |\delta\lambda_i| + \epsilon\hat{\kappa}\sqrt{1 + \left\|\hat{K}\right\|^2}. \tag{23}$$

Here κ, $\hat{\kappa}$ are the scaled spectral condition numbers of $A + BK$ and $\hat{A} + \hat{B}\hat{K}$, respectively and $B^\dagger$ is the Moore-Penrose pseudoinverse of B.

Note that under additional mild assumptions in the bounds (22) and (23) the terms $\hat{\kappa}, \hat{K}$ can be replaced by κ and K, respectively. If this is not possible, then the problem is extremely ill-conditioned and hence not suitable for numerical computation anyway.

Theorem 1 only gives upper bounds for the perturbations. This is the usual situation in most perturbation results. But these bounds are usually quite tight and very well describe the major difficulties of the pole placement problem. Consider the following numerical example from [61]. For this and all the other numerical examples the results were obtained on an HP-700 workstation with machine precision $eps = 2.22 \times 10^{-16}$, under MATLAB Version 5.2.

Example 1 Let $A = \mathrm{diag}(1, \ldots, 20)$, $\mathcal{P} = \{-1, \ldots, -20\}$ and let B be formed from the first m columns of a random 20×20 orthogonal matrix.

The MATLAB pole placement code *place* of the *control system toolbox* Version 4.1, which is an implementation of the method given in [43], was used to compute the feedback gain K. We ran m from 1 to 20 and in each case we computed 20 times with 20 random orthogonal matrices B. In Table 1 we list the geometric means (over the 20 experiments) of $\hat{\kappa}$, $\hat{K}$, bound=$eps\,\|[A,B]\|\,\hat{\kappa}\sqrt{1+\left\|\hat{K}\right\|^2}$, and err=$\max_{1\le i\le 20}|\mu_i-\lambda_i|$, with λ_i and the real parts of μ_i arranged in increasing order.

It should be noted that for all 400 tests the values of $\min_i \sigma_n([A-\lambda_i I, B])$ varied from 2.0 to 2.24, so the factor in the denominator of (22) is negligible. Furthermore we computed in all cases the distance to uncontrollability and found that the pair (A, B) was controllable with a large distance to uncontrollability. Nevertheless for $m = 1$ the method produced an error message "Can't place eigenvalues there" and for $m = 2, 3$ a warning "Pole locations are more than 10% in error" was displayed. The reason for this failure of the method is probably due to the large norm of K and the large closed loop condition number which is computed in the course of the algorithm. Other pole placement algorithms have similar difficulties for small m, see [60, 61].

m	$\hat{\kappa}$	$\hat{K}$	Bound	Err
1				
2	1.1×10^9	2.5×10^6	1.2×10^1	2.0×10^1
3	4.6×10^8	1.3×10^6	2.6	1.2×10^1
4	9.6×10^6	2.3×10^5	9.6×10^{-3}	1.2×10^{-3}
5	3.0×10^5	3.4×10^4	4.6×10^{-5}	1.6×10^{-6}
6	3.0×10^4	1.0×10^4	1.3×10^{-6}	3.1×10^{-8}
7	5.6×10^3	4.2×10^3	1.0×10^{-7}	1.3×10^{-9}
8	1.6×10^3	2.1×10^3	1.5×10^{-8}	1.3×10^{-10}
9	5.3×10^2	1.1×10^3	2.6×10^{-9}	1.9×10^{-11}
10	2.7×10^2	8.9×10^2	1.1×10^{-9}	6.3×10^{-12}
11	1.2×10^2	5.2×10^2	2.7×10^{-10}	1.8×10^{-12}
12	7.6×10^1	4.0×10^2	1.4×10^{-10}	8.3×10^{-13}
13	4.4×10^1	2.7×10^2	5.3×10^{-11}	3.6×10^{-13}
14	3.0×10^1	1.9×10^2	2.6×10^{-11}	2.0×10^{-13}
15	2.4×10^1	1.6×10^2	1.7×10^{-11}	1.5×10^{-13}
16	1.9×10^1	1.3×10^2	1.1×10^{-11}	9.5×10^{-14}
17	1.5×10^1	1.2×10^2	7.8×10^{-12}	6.9×10^{-14}
18	1.3×10^1	1.1×10^2	6.8×10^{-12}	6.6×10^{-14}
19	9.0	8.8×10^1	3.5×10^{-12}	4.5×10^{-14}
20	1.0	4.0×10^1	1.8×10^{-13}	3.2×10^{-14}

Table 1, results for Example 1.

The results of Example 1 and most other examples with $n - m$ large lead to the interpretation that the sensitivity (conditioning) of all possible results of the pole placement problem, i.e., the feedback gain K as well as the poles of the the closed-loop system $A+B\hat{K}$ obtained with the perturbed feedback $\hat{K}$, depends heavily on the size of $n - m$ as well as on the factor

$$\mathcal{S} := \kappa\sqrt{1 + \|K\|^2} \tag{24}$$

even if the distance to uncontrollability is large. The additional factor $d := 1/\min_i \sigma_n[A - \lambda_i I, B]$ in the perturbation bound only plays a role if the distance to uncontrollability is small. It is obvious that if $d_u(A, B)$ is small then d may be very large and the problem to compute K is definitely ill-conditioned. If, however, $d_u(A, B)$ is large, then clearly d is small and may be neglected.

The factor $\mathcal{S}$ has been analyzed in detail in [60, 61], where it was observed that in the single-input case $\mathcal{S}$ is essentially given by the condition number

of the Cauchy matrix $C = [\frac{1}{\nu_i - \lambda_j}]$, where the ν_i are the eigenvalues of A and the λ_i are the desired poles. This condition number is very large if n is large. In the multi-input case $\mathcal{S}$ is essentially given by the condition number of a Vandermonde-like matrix which is usually also very ill-conditioned (see [39, Chapter 21] and the references therein), in particular if $n - m$ is large.

This analysis indicates that serious numerical difficulties may arise in the pole placement problem if $n - m$ is large. Furthermore the analysis demonstrates that the currently used strategies to resolve the freedom in K in the numerical method, which is to minimize $\|K\|$, see [15, 44, 64, 67, 73, 83] or κ as in [43], may both not be sufficient to get good results. A better choice would be to minimize $\mathcal{S} := \kappa\sqrt{1 + \|K\|^2}$, since this factor describes the perturbation very well. A similar strategy has been proposed and implemented by Varga [84]. We can actually formulate this strategy as a refined **robust pole placement problem**.

Problem 2 *For given matrices $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$ and a given set of n complex numbers $\mathcal{P} = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$, (closed under conjugation), find a matrix $K \in \mathbb{R}^{m,n}$, such that the set of eigenvalues of $A + BK$ is equal to $\mathcal{P}$, and that minimizes $\mathcal{S} := \kappa\sqrt{1 + \|K\|^2}$.*

A solution to this problem for small systems can actually be obtained via standard optimization software by using the explicit formula for K given in [61]. In practice one probably does not even need the global minimum, but just one, where $\mathcal{S}$ is small enough to guarantee small bounds (22) and (23), which then can be actually computed and used as condition estimator.

But we propose to go even further in the reformulation of the pole placement problem, see also [36]. One should first ask the following question.

Does one really have a fixed set of poles or does one rather have a specific region in the complex plane where one wants the closed loop poles to be?

If the latter is the case then not only the minimization over the freedom in K but also a minimization over the position of the poles in the given set should be used. This would lead to the **optimized robust pole placement problem**:

Problem 3 *For given matrices $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$ and a given set $\mathcal{P} \subset \mathbb{C}$, find a matrix $K \in \mathbb{R}^{m,n}$, such that the set of eigenvalues of $A + BK$ is contained in $\mathcal{P}$ and at the same time a robustness measure is optimized.*

There are many papers that cover the placing of poles in specified regions like disks, strips or sectors, or the optimized placement of poles, see e.g. [14, 25, 40, 41, 50, 69, 72, 75, 78, 87]

and the references therein. A clear and practical formulation of such a general robust measure as well as suitable algorithms to determine this optimized pole assignment will depend on the application and on the set $\mathcal{P}$. In the stabilization problem this is the left half plane or in the case of damped stabilization a particular part of the left half plane, see [38]. If the set $\mathcal{P}$ is a very small region of the complex plane, as when it has exactly n points, then, as we have demonstrated above, even an optimization of some robustness measures may still yield a very sensitive system, but if the set $\mathcal{P}$ covers a large area in the complex plane, then quite good results may be obtained, see for example [23].

In the case of stabilization the robustness measure would certainly include the distance to instability, i.e., the smallest perturbation that makes the closed loop system have an unstable eigenvalue. To make sure that the closed loop system is securely stable, a constraint should be added in the optimization that guarantees that the perturbation bounds are smaller than the distance to instability. To verify and guarantee this constraint the distance to instability as well as the perturbation bound have to be computed, which alone is a difficult numerical problem, see [22]. In the context of stabilization this would be a part of the optimization loop and from this it may already be seen that the development of good numerical methods for this optimized stabilization is an important but extremely difficult problem that needs a lot of further attention, see also [62].

For large control problems with only few unstable poles the situation can be reduced to a small problem provided one can design a method for the separation of eigenvalues inside $\mathcal{P}$ and outside of $\mathcal{P}$. If this can be done, then the complexity of the optimization problem can be drastically reduced, see [71, 37, 83] and the references therein.

As we have mentioned already before, for the stabilization problem there are also other approaches to design a stabilizing feedback, such as the solution of Lyapunov or Riccati equations or just the solution of the linear quadratic control problem which we discuss in the next section. A comparison of stabilization via pole placement and linear quadratic control is given in Section 4.

3 Linear quadratic control

For the solution of the linear quadratic control problem, i.e., to minimize (4) subject to (1), a large number of approaches have been discussed in the literature, see the monographs [59, 66, 52, 74]. Let us compare the

Riccati equation approach with the solution of the two-point boundary value problem via a matrix pencil approach. An observation of Van Dooren [81] is that it suffices to study the deflating subspaces of the pencil $(\mathcal{E}_c, \mathcal{A}_c)$ in (6). Suppose $(\mathcal{E}_c, \mathcal{A}_c)$ has an n-dimensional deflating subspace associated with eigenvalues in the left half plane. Let this subspace be spanned by the columns of a matrix $\mathcal{U}$, partitioned analogous to the pencil as

$$\mathcal{U} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}. \tag{25}$$

Then, if U_1 is invertible, the optimal control is a linear feedback of the form $u(t) = U_3 U_1^{-1} x(t)$. The solution of the associated Riccati equation (9) is $X = U_2 U_1^{-1}$, see [59] for details. We see that an explicit solution of the Riccati equation is not needed to determine the optimal control and it is also clear that the sensitivity of the computation of $U_3 U_1^{-1} x(t)$ may be different than that of the procedure to first compute $X = U_2 U_1^{-1}$ and then the feedback $u(t) = -R^{-1} B^T X x(t)$ from this. In particular if the matrix R is close to singular, then the coefficients in the Riccati equation (9) may be highly corrupted so that a solution approach via the Riccati equation may be completely useless. We demonstrate these observations in the following example.

Example 2 Let U be a randomly generated real orthogonal matrix, $L = 0$, $A = U \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} U^T$, $B = U$, $R = \begin{bmatrix} 0.5 & 0 \\ 0 & \gamma \end{bmatrix}$ and $Q = U \begin{bmatrix} 6 & 0 \\ 0 & 3\gamma \end{bmatrix} U^T$, where $\gamma > 0$.

The positive semidefinite (stabilizing) solution of the corresponding algebraic Riccati equation (9) is $X = U \begin{bmatrix} 3 & 0 \\ 0 & 3\gamma \end{bmatrix} U^T$, the associated feedback gain matrix $K = -\begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix} U^T$ and the closed loop spectrum is $\{-4, -2\}$, both independent of the value of γ. Since U is orthogonal, we see that $\|K\|$ is small and hence we do not expect large perturbations in the solution. The solution via the Riccati equation, however, depends on γ and hence we may expect that the feedback K when computed via the Riccati equation will depend heavily on γ.

We applied the MATLAB m-files *are*, *care* from different versions of the MATLAB control tool box [55] which are solvers for algebraic Riccati equations and compare the results with those obtained by just computing the deflating subspace by the MATLAB implementation *qz* of the QZ-algorithm.

The Riccati solution is used to compute $K = -R^{-1}B^TX$ while via the deflating subspace (25) of $\alpha\mathcal{E}_c - \beta\mathcal{A}_c$, the feedback K is directly obtained as $U_3U_1^{-1}$. The method *are* uses the Hamiltonian matrix $\mathcal{H}$ as in (8) to determine the Riccati solution X while the method *care* works on a balanced version of $\mathcal{H}$ if $\frac{\lambda_{\min}(R)}{\lambda_{\max}(R)} \geq \sqrt{eps}$ and on the extended pencil $\alpha\mathcal{E}_c - \beta\mathcal{A}_c$ as in (6) otherwise.

The relative error in X and K for all three methods and different values of γ are listed in Table 2.

γ	Method	$\frac{\|\hat{X}-X\|_2}{\|X\|_2}$	$\frac{\|\hat{K}-K\|_2}{\|K\|_2}$
10^{-2}	are	7.6×10^{-16}	2.1×10^{-14}
	care	7.0×10^{-16}	1.3×10^{-15}
	qz	2.4×10^{-16}	4.9×10^{-15}
10^{-6}	are	3.5×10^{-11}	5.7×10^{-7}
	care	3.1×10^{-12}	3.2×10^{-9}
	qz	2.6×10^{-15}	4.7×10^{-11}
10^{-9}	are	1.8×10^{-8}	9.1×10^{-1}
	care	2.1×10^{-8}	1.3×10^{-4}
	qz	1.6×10^{-15}	5.9×10^{-9}
10^{-13}	are	7.7×10^{-5}	1.2×10^{4}
	care	9.2×10^{-5}	3.9×10^{1}
	qz	1.7×10^{-15}	5.0×10^{-4}

Table 2, Relative errors in Example 2.

We see that the direct computation of the optimal control via the subspace yields much smaller relative errors than the solution via the Riccati equation. Note that the subspace method always computed the Riccati solution to high relative accuracy.

This example demonstrates that the solution of the linear quadratic control problem via the solution of the algebraic Riccati equation presents a dangerous detour that may lead to very bad results and is really not necessary, since the feedback and the closed loop matrix can be computed from the deflating subspace of the extended pencil directly. This is even more critical in the situation that R is indefinite or singular as in the H_∞ problem discussed below. The situation is even worse in the case of descriptor systems, see [8, 9, 59], where it is known that the Riccati equation may not have anything to do with the solution of the optimal control problem [49].

But also for the linear quadratic control problem the question of robustness has to be asked in terms of the performance criterion, i.e., the choice

of Q, L, R which, as we have seen in Example 2, is critical in the Riccati approach. But since this is a freedom in the problem, we should make use of it to optimize the robustness. In the context of stabilization or other regions $\mathcal{P}$ of the complex plane we may, therefore, formulate the **optimized linear quadratic control problem.**

Problem 4 *Consider matrices $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$ and a set $\mathcal{P} \subset \mathbb{C}$. Determine cost matrices Q, L, R such the the closed loop system obtained via the solution of the associated linear quadratic control problem has eigenvalues that are contained in $\mathcal{P}$ and at the same time a robustness measure is optimized.*

If the robustness measure in Problem 4 is the same as in Problem 3, then these two problems are actually equivalent.

Proposition 2 *Consider matrices $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$ and a set $\mathcal{P} \subset \mathbb{C}$. Consider furthermore the optimized linear quadratic control problem 4 and the optimized robust pole assignment problem 3. If the same robustness measure is used in both problems, then the problems are equivalent, i.e., they have the same solution sets.*

Proof. Since the feedbacks in Problem 3 are not restricted, it is clear that the solution set of Problem 3 contains the solution set of Problem 4. Suppose now that a feedback gain K optimizes Problem 3. Choosing an arbitrary positive definite matrix R and setting $L = -K^T R$, $Q = LR^{-1}L^T$, it follows that the linear quadratic control generates the same feedback gain matrix K as well as the same closed-loop system $A+BK$. Hence the solution set of Problem 3 is contained in the solution set of Problem 4. □

It should be noted, however, that in many applications cost functionals with $L = 0$ are used. In this situation the optimal solution via Problem 4 may be worse than that of Problem 3 as the following example demonstrates, see also Example 4.

Example 3 Consider the scalar system with $A = 1$ and $B = 1$ and the set $\mathcal{P} = \{x | \operatorname{Re} x \leq -\alpha,\ 0 < \alpha < 1\}$. Obviously in this case the distance to uncontrollability satisfies $d_u(A, B) = 1$, and the scaled spectral condition is $\kappa(A + BK) = 1$ for arbitrary K. Thus we only need to minimize $\|K\|_2$. For Problem 3 the optimal feedback is $K = -(1+\alpha)$ and the closed loop system is $A + BK = -\alpha$. However, for Problem 4 with $L = 0$, the optimal solution, i.e., the minimum norm K, is $K = -2$ which is obtained with arbitary $R > 0$ and $Q = 0$. The associated closed loop system is $A + BK = -1$. In fact

for $R > 0$ and $Q \geq 0$ the pole of $A + BK$ is $-\sqrt{1 + Q/R}$ which cannot be greater than -1.

It follows from this example that in order to obtain results which are as good as those from optimized robust pole placement the block L in the cost functional has to be included in the optimization.

As we have discussed already in the context of pole assignment, there are many different possibilities of general robust measures. These depend on the specific application and lead to different numerical methods. An analysis of different criteria should deserve more attention. Some numerical examples in the context of stabilization are discussed in the next section.

4 Stabilization

In this section we compare the results obtained from optimized robust pole assignment and optimized linear quadratic control for the specific problem of stabilization, i.e., the set $\mathcal{P}$ is the open left half plane.

Our first example discusses the optimization of the condition number $\mathcal{S}$ in (24) in the particular situation that in the cost functional we use $L = 0$.

Example 4 Consider the stabilization problem with $A = \mathrm{diag}(1, 2, 3, 4)$ and $B = [1, 1, 1, 1]^T$ and a stability margin of 0.5, i.e., $\mathcal{P} = \{\lambda \in \mathbb{C} | \, \mathrm{Re}\,(\lambda) \leq -0.5\}$.

We used a heuristic 'random search' algorithm for the optimal poles as in [62], to minimize the condition number $\mathcal{S}$ in (24). For the solution of the pole placement problem a MATLAB code based on the method of Miminis and Paige [64] was used. It should be noted that the MATLAB code *place* often generated incorrect results, which is probably due to a small distance to instability in some of the cases. The computed optimal poles, the norm of the feedback gain and the condition number $\mathcal{S}$ are listed in Table 3, as well as the distance to instability displayed in column *dis* of the closed loop matrix $A + BK$. The distance to instability was computed by the method of Byers [22].

For comparison we used the solution of the optimized linear quadratic control problem with a shift, see e.g. [38], to compute the feedback gain using the MATLAB code *surv* based on the structure preserving Algorithm 1 below to determine the feedback gains. In the cost functional we chose $L = 0$ and $R = 50 * \|B\|^2 / k$ with $k = 1, \ldots, 100$ as well as $R = \|B\|^2 / 2^{k+1}$ with $k = 1, \ldots, 20$. For each such R we chose 100 randomly chosen unit norm positive definite matrices Q. Note that, as desired, all eigenvalues of

$A + BK$ have real parts less than -0.5. Among all tests the minimum for $\mathcal{S}$ was obtained for $R = 1/2^6$ (note $\|B\| = 2$). The results are also shown in Table 3.

Method	closed loop poles	$\|K\|$	$\mathcal{S}$	dis
Pole placement	$-0.5 \pm 3.69i$, $-0.5 \pm 1.02i$	222	1.1×10^5	0.005
LQ	$-12.6, -4.26, -3.04, -1.66$	2.0×10^3	3.9×10^7	0.013

Table 3, a comparison between stabilization by LQ and pole placement.

We see from this example, as we have already discussed before, that optimized robust pole assigment performs better than optimized linear quadratic control with $L = 0$. On the other hand even for this small sized single input problem the optimal condition number is very large.

Furthermore we observe and this is typical, see also [62], that the optimal condition number is obtained with eigenvalues close to or on the boundary of the desired region. Thus if we choose the region $\mathcal{P}$ to be the open left half plane then we will typically get a small distance to instability. For this reason and to show that more theoretical investigation is necessary, in the next example we compare different optimality criteria.

Example 5 Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $B = I_2$ and $\mathcal{P} = \{\lambda \in \mathbb{C} | \operatorname{Re}(\lambda) \leq -1\}$. As robustness measures we minimize κ_F, $\|K\|_F$ and $\mathcal{S}_F = \kappa_F \sqrt{1 + \|K\|_F^2}$, respectively, where the index F indicates that the Frobenius norm is used. Clearly in this case $K = T\Lambda T^{-1} - A$ for an arbitrary nonsingular real matrix T and arbitrary real Λ with eigenvalues in the required region.

If the scaled spectral condition number of the closed loop system is to be minimized, then the optimal solution is obtained with an orthogonal matrix T and freely chosen Λ.

In the optimization of $\|K\|_F$ and $\mathcal{S}_F$ the optimal case is that Λ has a pair of complex conjugate eigenvalues. Let $\Lambda = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$. The general form of T is $T = \gamma T_s \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$, where $\gamma, a \neq 0$ and T_s is a plane rotation. Since T_s commutes with Λ and since γ does not affect the norms, we can set $T_s = I_2$ and $\gamma = 1$. To simplify the computation of the minimal $\mathcal{S}_F$ we furthermore set $b = 0$, which only gives a suboptimal result. In Table 5 we give the resulting values of $\mathcal{S}_F, \|K\|_F$ as well as the distance to instability *dis* of the associated closed loop matrix $A + BK$. Here in the optimization of κ_F we have chosen both eigenvalues to be at -1.

objective	closed loop poles	$\|F\|_F$	κ_F	$\mathcal{S}_F$	*dis*
κ_F	$\{-1\}$	3.74	2	7.75	1.0
$\|F\|_F$	$-1 \pm 0.5 \times 10^{-8} i$	3.54	2.4×10^8	8.7×10^8	0.56
$\mathcal{S}_F$	$-1 \pm 0.52i$	3.67	2.001	7.61	0.9994

Table 4, a comparison of optimality criteria.

The associated feedback gain matrices in the three cases are

$$-\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, \quad -\begin{bmatrix} 2.4534 & 0 \\ 0.2056 & 2.5466 \end{bmatrix}, \quad -\begin{bmatrix} 2 & 0.4656 \\ 0.4988 & 3 \end{bmatrix},$$

respectively.

We see from this example that a pure optimization of $\|K\|_F$ may lead to drastically different results than an optimization of κ_F and $\mathcal{S}_F$ but we also see that a detailed further investigation is necessary to obtain the best possible criteria.

5 Structure preservation

In the context of the linear quadratic control problem the second important topic that needs to be discussed, is the preservation of structure.

A feature of the pencils associated with the two-point boundary value problem (5) is that they have algebraic structures which lead to a certain symmetry in the spectrum. Roundoff errors can destroy this symmetry leading to physically meaningless results unless the numerical method also preserves the algebraic structure, see [80]. Moreover, preservation of the algebraic structure usually leads to more efficient as well as more accurate numerical methods. Let us briefly introduce the relevant structures.

Definition 3 *Let* $J := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, *where* I_n *is the* $n \times n$ *identity matrix.*

a) A matrix $\mathcal{H} \in \mathbf{R}^{2n\times 2n}$ *is* Hamiltonian *if* $(\mathcal{H}J)^T = \mathcal{H}J$ *and a matrix* $\mathcal{H} \in \mathbf{R}^{2n\times 2n}$ *is* skew-Hamiltonian *if* $(\mathcal{H}J)^T = -\mathcal{H}J$.

b) A matrix $\mathcal{Z} \in \mathbf{R}^{n\times n}$ *is* symplectic *if* $\mathcal{Z}J\mathcal{Z}^T = J$ *and a matrix* $\mathcal{U} \in \mathbf{R}^{2n\times 2n}$ *is* orthogonal symplectic *if* $\mathcal{U}J\mathcal{U}^T = J$ *and* $\mathcal{U}\mathcal{U}^T = I_{2n}$. *The group of orthogonal symplectic matrices in* $\mathbf{R}^{n\times n}$ *is denoted by* $\mathcal{US}_{2n}$.

c) *We call a real matrix* Hamiltonian quasi-triangular *if it is Hamiltonian and has the form*

$$\begin{bmatrix} F & G \\ 0 & -F^T \end{bmatrix},$$

where F is quasi-triangular in real Schur form, see [33]. If a Hamiltonian matrix $\mathcal{H}$ can be transformed into Hamiltonian quasi-triangular form by a similarity transformation with a matrix $\mathcal{U} \in \mathcal{US}_{2n}$, then we say that $\mathcal{U}^T\mathcal{H}\mathcal{U}$ has Hamiltonian Schur form.

The reduced Euler-Lagrange equations (7) involve a Hamiltonian matrix, but the pencil (6) does not directly have this structure. Nonetheless many of the properties of Hamiltonian matrices carry over, see [59]. Furthermore, we may endow the pencil (6) with a similar structure by embedding the Euler-Lagrange equations (5) into a larger system. If m is even then this is easily done by splitting $u(t), B, L, R$ into half sized parts and a permutation of the pencil, see [8]. If m is odd then we may apply this splitting after introducing an artificial input. The resulting pencil (after some permutation) has the form

$$\alpha\mathcal{E}_c^e - \beta\mathcal{A}_c^e := \alpha \left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] - \beta \left[\begin{array}{cc|cc} A & B_1 & 0 & B_2 \\ L_2^H & R_{12}^H & B_2^H & R_{22} \\ \hline -Q & -L_1 & -A^H & -L_2 \\ -L_1^H & -R_{11} & -{B_1}^H & -R_{12} \end{array}\right]. \tag{26}$$

with one Hamiltonian and one skew-Hamiltonian matrix.

The solution of the eigenproblem for Hamiltonian matrices and skew-Hamiltonian/Hamiltonian pencils has been a topic of several publications, see [8, 18, 53, 57, 58, 59] and the references therein. The goal is to obtain a numerically backward stable method, that has a complexity of $\mathcal{O}(n^3)$ and at the same time preserves the structure. There are two main reasons why this problem is difficult. First of all one needs a triangular-like form under orthogonal symplectic similarity transformations from which the desired invariant subspaces can be read off. Such a Hamiltonian Schur form was first suggested in [65] but not every Hamiltonian matrix or skew-Hamiltonian/Hamiltonian pencil has such a condensed form, see [54, 57, 58]. The second difficulty arises from the fact that even if a Hamiltonian Schur form exists, it is still difficult to construct a method with the desired features, see [2, 3, 9, 10, 20, 21].

We dicuss here only the computation of the structured Schur form for Hamiltonian matrices. For skew-Hamiltonian/Hamiltonian pencils we refer

the reader to [9, 57, 58]. Necessary and sufficient conditions for the Hamiltonian Schur form are given by the following theorem.

Theorem 4 *[54] Let $\mathcal{H}$ be a real Hamiltonian matrix, let $i\alpha_1, \ldots, i\alpha_\nu$ be its pairwise distinct nonzero purely imaginary eigenvalues and let U_k, $k = 1, \ldots, \nu$, be the associated invariant subspaces. Then the following are equivalent.*

i) There exists a real symplectic matrix $\mathcal{Z}$ such that $\mathcal{Z}^{-1}\mathcal{H}\mathcal{Z}$ is real Hamiltonian quasi-triangular.

ii) There exists a real orthogonal symplectic matrix $\mathcal{U}$ such that $\mathcal{U}^T\mathcal{H}\mathcal{U}$ is real Hamiltonian quasi-triangular.

iii) $U_k^H J U_k$ is congruent to J for all $k = 1, \ldots, \nu$, where J is always of the appropriate dimension.

A similar theorem for skew-Hamiltonian/Hamiltonian pencils has been given in [57, 58].

This result shows that whenever a structured triangular form exists, then it also exists under orthogonal transformations and hence there is hope that these forms and therefore also the eigenvalues and invariant and deflating subspaces can be computed with structure preserving numerically stable methods.

Let us first discuss the computation of eigenvalues. It is well-known that if $\mathcal{H}$ is a Hamiltonian matrix, then $\mathcal{H}^2$ is a skew-Hamiltonian matrix for which a structure preserving method was suggested in [82]. This suggests computing the eigenvalues of $\mathcal{H}$ by taking square roots of the eigenvalues of $\mathcal{H}^2$. Unfortunately, in a worst case scenario via this approach one might obtain only half of the possible accuracy in the computed eigenvalues [20, 82]. A way out of this dilemma was recently presented in [11]. This approach uses the following decomposition.

Theorem 5 *[11] Let $\mathcal{H}$ be Hamiltonian. Then there exist $Q_1, Q_2 \in \mathcal{US}_{2n}$, such that*

$$Q_1^T \mathcal{H} Q_2 = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix}, \tag{27}$$

with H_{11} upper triangular and H_{22}^T quasi upper triangular. Furthermore the eigenvalues of $\mathcal{H}$ are the square roots of the eigenvalues of $-H_{11}H_{22}^T$.

Note that the resulting matrix in (27) is neither Hamiltonian nor similar to $\mathcal{H}$, but a simple calculation shows that both $Q_1^T\mathcal{H}^2Q_1$ and $Q_2^T\mathcal{H}^2Q_2$ are real skew-Hamiltonian quasi-triangular. For skew-Hamiltonian/Hamiltonian pencils similar results have been given in [9]. After the form (27) has been computed, one can compute the eigenvalues of $\mathcal{H}$ by solving 1×1 or 2×2 eigenvalue problems and taking square roots without loosing accuracy. For algorithmic details, a detailed error analysis as well as illustrative numerical examples, see [11], where it is demonstrated that these methods speed up the computation of eigenvalues while still achieving full possible accuracy.

This new approach has also been extended to the computation of the desired deflating and invariant subspaces. Let us first introduce the basic theory behind the method. Let for $A\in\mathbf{R}^{n\times n}$ the sets $\lambda_-(A),\lambda_+(A),\lambda_0(A)$ denote the part of the spectrum of A in the open left half plane, in the open right half plane and on the imaginary axis, respectively and denote the associated invariant subspaces by $\mathrm{Inv}_-(A),\mathrm{Inv}_+(A)$, $\mathrm{Inv}_0(A)$. In [10] it has been observed that for $A\in\mathbf{R}^{n\times n}$ and $B=\begin{bmatrix} 0 & A\\ A & 0\end{bmatrix}$, if one determines an orthogonal matrix such that

$$B\begin{bmatrix} Q_1\\ Q_2\end{bmatrix}=\begin{bmatrix} Q_1\\ Q_2\end{bmatrix}R, \tag{28}$$

where

$$\lambda_+(B)\subseteq\lambda(R)\subseteq\lambda_+(B)\cup\lambda_0(B), \tag{29}$$

then

$$\mathrm{range}\{Q_1+Q_2\}=\mathrm{Inv}_+(A)+\mathcal{N}_1,\quad \text{where}\quad \mathcal{N}_1\subseteq\mathrm{Inv}_0(A), \tag{30}$$

$$\mathrm{range}\{Q_1-Q_2\}=\mathrm{Inv}_-(A)+\mathcal{N}_2,\quad \text{where}\quad \mathcal{N}_2\subseteq\mathrm{Inv}_0(A). \tag{31}$$

Moreover, if we partition $R=\begin{bmatrix} R_{11} & R_{12}\\ 0 & R_{22}\end{bmatrix}$ with $\lambda(R_{11})=\lambda_+(B)$ and, accordingly, $Q_1=\begin{bmatrix} Q_{11} & Q_{12}\end{bmatrix}$, $Q_2=\begin{bmatrix} Q_{21} & Q_{22}\end{bmatrix}$, then

$$B\begin{bmatrix} Q_{11}\\ Q_{21}\end{bmatrix}=\begin{bmatrix} Q_{11}\\ Q_{21}\end{bmatrix}R_{11}, \tag{32}$$

and there exists an orthogonal matrix Z such that

$$\begin{aligned}\frac{\sqrt{2}}{2}(Q_{11}+Q_{21}) &= \begin{bmatrix} 0 & P_+\end{bmatrix}Z,\\ \frac{\sqrt{2}}{2}(Q_{11}-Q_{21}) &= \begin{bmatrix} P_- & 0\end{bmatrix}Z,\end{aligned} \tag{33}$$

where P_+, P_- are orthogonal bases of $\mathrm{Inv}_+(A)$, $\mathrm{Inv}_-(A)$, respectively.

In the case of a Hamiltonian matrix $\mathcal{H} = \begin{bmatrix} F & G \\ H & -F^T \end{bmatrix}$ one considers the block matrix $\mathcal{B} = \begin{bmatrix} 0 & \mathcal{H} \\ \mathcal{H} & 0 \end{bmatrix}$ and, using the block permutation $\mathcal{P} = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix}$, one obtains that

$$\tilde{\mathcal{B}} := \mathcal{P}^T \mathcal{B} \mathcal{P} = \begin{bmatrix} 0 & F & 0 & G \\ F & 0 & G & 0 \\ 0 & H & 0 & -F^T \\ H & 0 & -F^T & 0 \end{bmatrix} \tag{34}$$

is again Hamiltonian. Furthermore it follows from Theorem 4 that $\tilde{\mathcal{B}}$ has a Hamiltonian Schur form.

Theorem 6 *[10] Let $\mathcal{H}$ be Hamiltonian and let $\mathcal{B} = \begin{bmatrix} 0 & \mathcal{H} \\ \mathcal{H} & 0 \end{bmatrix}$. Then there exists an orthogonal matrix $\mathcal{U}$ such that*

$$\mathcal{U}^T \mathcal{B} \mathcal{U} = \begin{bmatrix} R & D \\ 0 & -R^T \end{bmatrix} =: \mathcal{R} \tag{35}$$

is in Hamiltonian quasi-triangular form and $\lambda_-(R) = \emptyset$. Moreover, $\mathcal{U} = \mathcal{P}\mathcal{W}$ with $\mathcal{W} \in US_{4n}$, and

$$\mathcal{R} = \mathcal{W}^T \tilde{\mathcal{B}} \mathcal{W}, \tag{36}$$

i.e., $\mathcal{R}$ is the Hamiltonian quasi-triangular form of the Hamiltonian matrix $\tilde{\mathcal{B}}$. Furthermore, if $\mathcal{H}$ has no purely imaginary eigenvalues, then R has only eigenvalues with positive real part.

The structure preserving, numerically stable algorithm to compute the invariant subspace of a Hamiltonian matrix associated with the eigenvalues in the left half plane is then as follows.

Algorithm 1

Input: *A Hamiltonian matrix $\mathcal{H}$ having an n-dimensional Lagrangian invariant subspace associated with the eigenvalues in the left half plane.*

Output: *$Y \in \mathbf{R}^{2n\times n}$, with $Y^TY = I_n$, such that the columns of Y span this invariant subspace.*

Step 1 *Apply Algorithm 2 of [11] to $\mathcal{H}$ and compute orthogonal symplectic matrices $Q_1, Q_2 \in \mathcal{US}_{2n}$ such that*

$$Q_1^T\mathcal{H}Q_2 = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix}$$

is the decomposition (27).

Step 2 *Determine an orthogonal matrix Q_3, such that*

$$Q_3^T \begin{bmatrix} 0 & -H_{22}^T \\ H_{11} & 0 \end{bmatrix} Q_3 = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}$$

is in real Schur form ordered such that the eigenvalues of T_{11} have positive real part and the eigenvalues of T_{22} have negative real part.

Step 3 *Use the orthogonal symplectic reordering scheme of [21] to determine an orthogonal symplectic matrix $V \in \mathcal{US}_{4n}$ such that with*

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} := \begin{bmatrix} Q_1Q_3 & 0 \\ 0 & Q_2Q_3 \end{bmatrix} V$$

we have the Hamiltonian quasi-triangular form

$$U^T\mathcal{B}U = \begin{bmatrix} F_{11} & F_{12} & G_{11} & G_{12} \\ 0 & F_{22} & G_{21} & G_{22} \\ 0 & 0 & -F_{11}^T & 0 \\ 0 & 0 & -F_{12}^T & -F_{22}^T \end{bmatrix},$$

where F_{11}, F_{22} are quasi upper triangular with eigenvalues only in the closed right half plane.

Step 4 *Set $\hat{Y} := \frac{\sqrt{2}}{2}(U_{11} - U_{21})$. Compute Y, an orthogonal basis of* range$\{\hat{Y}\}$, *using any numerically stable orthogonalization scheme, for example a rank-revealing QR-decomposition; see, e.g., [24].*

End

Generalizations of these results to the complex case and algorithms are presented in [12]. Corresponding results and methods for skew-Hamiltonian/Hamiltonian pencils have been constructed in [9].

It should be noted that these new methods are already very close to the desired structure preserving methods but they are still not optimal, since not all structures are fully exploited. But the methods are more efficient and at least as accurate as methods that do not address structure preservation. This approach works in principle also for Hamiltonian matrices with eigenvalues on the imaginary axis provided the appropriate subspaces can be seperated. When this is the case and how the numerical method can detect this, as well as the perturbation analysis is still under investigation, see [63]. A complete analysis of this case will be also very important for the treatment of H_∞ control problems, that we discuss in the next section.

6 Standard H_∞ control

The solution of the standard H_∞ control problem addresses another robustness measure in the computation of a feedback solution, which is different from the criteria that we have discussed so far. For the numerical solution of the H_∞ control problem the usual procedure is to use a optimization scheme to determine the smallest $\gamma > 0$ so that all three conditions (A1), (A2) and (A3) in Section 1 hold by determining the first value of γ where one of these conditions fail, see for example [34, 68, 88]. In each step of the optimization procedure two linear quadratic optimal control problems are solved plus a positivity check.

Typically in current design packages like the MATLAB robust control toolbox [56], the solution is obtained by a procedure which uses the solution of algebraic Riccati equations to determine X_∞ and Y_∞.

In view of the discussion in Section 3 on the solution of linear quadratic control problems and Riccati equations we should construct new methods for the H_∞ control problem that avoid the detour via the Riccati equation. This conclusion is complemented by the observation that during the optimization procedure, typically one or both of the Riccati solutions becomes very large in norm. This leads to the question whether a numerical solution of the H_∞ via the solution of Riccati equations makes sense at all, since in order to obtain a robust control, a highly ill-conditioned numerical problem has to be solved.

The usual way out of this dilemma in practice is to compute suboptimal controls, see [35, 68]. But in view of the previous discussions one might ask

whether this potential ill-conditioning is inherent in the problem formulation or due to the approach for its solution. Let us consider an example.

Example 6 *Let $A = 1$, $B_1 = 2$, $B_2 = 1$, $C_1 = 1$ and $C_2 = \sqrt{3}$. Then for $\gamma > \gamma_{x,1} = \sqrt{2}$ the matrix $\mathcal{H}_\infty$ in (13) has no purely imaginary eigenvalues and hence a Lagrange subspace associated with the stable eigenvalues always exists. The stabilizing solution of the Riccati equation, however, is $X(\gamma) = \frac{\gamma^2+\gamma\sqrt{2\gamma^2-4}}{\gamma^2-4}$. For $\gamma > \gamma_{x,2} = 2$ we have that $X(\gamma)$ is positive definite and for $\gamma < \gamma_{x,2}$, $X(\gamma)$ is negative definite. For $\gamma = \gamma_{x,2}$ the Riccati solution is not defined.*

Analogously for the Riccati equation associated with $\mathcal{J}_\infty$ in (15) we have $\gamma_{y,1} = \frac{2\sqrt{13}}{13}$ and $\gamma_{y,2} = \frac{\sqrt{3}}{3}$, and the associated stabilizing solution of the Riccati equation is $Y(\gamma) = \frac{\gamma^2+\gamma\sqrt{13\gamma^2-4}}{3\gamma^2-1}$. It follows that the optimal parameter γ_{opt} must be greater than $\gamma_{x,2} = 2$.

For the third condition (A3) we have $\gamma_{x,1}^2 > \rho(X(\gamma_{x,1})Y(\gamma_{x,1}))$, since $X(\gamma_{x,1}) = -1$ and $Y(\gamma_{x,1}) = \frac{2(1+\sqrt{11})}{5}$. But $\gamma_{x,1}$ is obviously not optimal. So in a typical optimization procedure to determine the optimal γ one needs first to determine $\gamma_{x,2}$ and $\gamma_{y,2}$, but $X(\gamma_{x,2}), Y(\gamma_{y,2})$ are not defined.

We see from this example that, as for the solution of the linear quadratic control problem, the Riccati solutions X_∞ and Y_∞ should be avoided. Fortunately this can again be done quite easily. In [88, Theorem 16.16, p. 445] it is shown that conditions (A1) – (A3) may be replaced by the alternative conditions

(B1) There exist matrices $Q_1, Q_2 \in \mathbb{R}^{n,n}$ such that

$$\mathcal{H}_\infty \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} T_x,$$

where T_x has only eigenvalues with nonpositive real parts and $Q_1^T Q_2 = Q_2^T Q_1$.

(B2) There exist matrices $U_1, U_2 \in \mathbb{R}^{n,n}$ such that

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}^T \mathcal{J}_\infty = T_y \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}^T,$$

where T_y has only eigenvalues with nonpositive real parts and $U_1^T U_2 = U_2^T U_1$.

(B3) $\begin{bmatrix} Q_2^T Q_1 & \gamma^{-1} Q_2^T U_2 \\ \gamma^{-1} U_2^T Q_2 & U_2^T U_1 \end{bmatrix}$ is symmetric positive semidefinite.

If these conditions hold then $\|T_{zw}\|_\infty \leq \gamma$ and the admissable controller is in descriptor form

$$\begin{aligned} \hat{E}\dot{q} &= \hat{A}q + \hat{B}y \\ u &= \hat{C}q + \hat{D}y, \end{aligned} \tag{37}$$

with $\hat{E} = U_1^T Q_1 - \gamma^{-1} U_2^T Q_2$, $\hat{B} = U_2^T C_2^T$, $\hat{C} = -B_2^T Q_2$, $\hat{D} = 0$ and $\hat{A} = \hat{E}T_x - \hat{B}C_2 Q_1 = T_y \hat{E} + U_1^T B_2 \hat{C}$.

Using this result, only the invariant subspaces of $\mathcal{H}_\infty$ and $\mathcal{J}_\infty$ are involved and they can be determined via the same methods that we have discussed in the previous section.

Thus not only is it possible to avoid the ill-conditioned Riccati equation but also we can employ structure preservation as described above and as in the case of the linear quadratic control problem, the computation of these subspaces is usually much better conditioned than the computation of the Riccati solutions.

Thus, the solution of the H_∞ control problem should be approached via the usual optimization procedures like in [19, 32, 35, 76], using in each optimization step Algorithm 1 to determine the subspaces in (B1) and (B2) and a Cholesky factorization to check condition (B3). An implementation and analysis of such a procedure is currently under investigation.

7 Conclusion

We have discussed several standard problems of linear control theory, like pole assignment, stabilization, linear quadratic and H_∞ control and have demonstrated some of the difficulties that arise in the numerical solution of these problems due to inherent ill-conditioning in the problem. We have also suggested several reformulated versions of the problem, which are sometimes more complicated to solve, but which yield results that are much more robust to perturbations.

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