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Volker Mehrmann   Hongguo Xu

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Volker Mehrmann* Hongguo Xu*

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Abstract

The existence and uniqueness of Lagrangian invariant subspaces of Hamiltonian matrices is studied. Necessary and sufficient conditions are given in terms of the Jordan structure and certain sign characteristics that give uniqueness of these subspaces even in the presence of purely imaginary eigenvalues. These results are applied to obtain in special cases existence and uniqueness results for Hermitian solutions of continuous time algebraic Riccati equations.

Keywords. eigenvalue problem, Hamiltonian matrix, symplectic matrix, Lagrangian invariant subspace, algebraic Riccati equation, structure inertia index
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1 Introduction

The solution of continuous time algebraic Riccati equations or the related problem of computing Lagrangian invariant subspaces is an important task in many applications in control, Kalman filtering , or spectral factorization [8, 14, 19].

Definition 1 *A $2n \times 2n$ complex matrix $\mathcal{H}$ is called Hamiltonian if $J_n\mathcal{H} = (J_n\mathcal{H})^H$ is Hermitian, where $J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, I_n is the $n \times n$ identity matrix and the superscript H denotes the conjugate transpose.*

Every Hamiltonian matrix $\mathcal{H}$ has the block form

$$\mathcal{H} = \begin{bmatrix} A & D \\ G & -A^H \end{bmatrix}, \tag{1}$$

*Fakultät für Mathematik, TU Chemnitz, D-09107 Chemnitz, FR Germany. This work was supported by *Deutsche Forschungsgemeinschaft*, Research Grant Me 790/7-2.

with $D = D^H, G = G^H$ and the related algebraic Riccati equation is

$$A^H X + XA - XDX + G = 0. \tag{2}$$

An immediate and well-known connection [14], between the invariant subspaces of a Hamiltonian matrix (1) and the Hermitian solutions of (2) is that if $X = X^H$ solves (2) then

$$\mathcal{H}\begin{bmatrix} I_n & 0 \\ -X & I_n \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ -X & I_n \end{bmatrix}\begin{bmatrix} (A-DX) & D \\ 0 & -(A-DX)^H \end{bmatrix}. \tag{3}$$

This implies that range $\begin{bmatrix} I_n \\ -X \end{bmatrix}$ is an invariant subspace of $\mathcal{H}$ associated with the eigenvalues of $A - DX$. Invariant subspaces that can be transformed to this form are called *graph subspaces* [14]. The graph subspaces of Hamiltonian matrices are special *Lagrangian subspaces.*

Definition 2 *A subspace $\mathcal{L}$ of $\mathbf{C}^{2n}$ is is called Lagrangian subspace if it has dimension n and*

$$x^H J_n y = 0, \quad \forall x, y \in \mathcal{L}.$$

The following result is well-known, see, e.g., [14].

Proposition 1 *The algebraic Riccati equation (2) has an Hermitian solution if and only if there exists a $2n \times n$ matrix $L := \begin{bmatrix} U \\ V \end{bmatrix}$ with $U \in \mathbf{C}^{n\times n}$ invertible, such that the columns of L span a Lagrangian invariant subspace of the related Hamiltonian matrix $\mathcal{H}$. In this case $X = -VU^{-1}$ is Hermitian and solves (2).*

It follows that we can study the existence and uniqueness of solutions of algebraic Riccati equations via the analysis of Lagrangian invariant subspaces of the associated Hamiltonian matrices. While for most classical situations the theory and also numerical solution methods are well established, [14, 19], there are still some long term open problems related to the case that the Hamiltonian matrix associated with the Riccati equation has purely imaginary eigenvalues. In this paper we will be mainly concerned with the characterization of the Lagrangian invariant subspaces rather than the solutions of the Riccati equation. The concept of Lagrangian invariant subspaces is a more general concept than that of Hermitian solutions of the Riccati equation, since for these subspaces we do not have the restriction that U is nonsingular. The characterization under which conditions every Lagrangian invariant subspace is a graph subspace, i.e., is associated with a solution of the algebraic Riccati equation has recently been given in [1, 11].

Another reason for the importance of the Lagrangian invariant subspaces is that most numerical solution methods (with the exception of Newton's method) proceed via the computation of the Lagrangian invariant subspaces to determine the solution of the Riccati equation or directly the solution of the control problems, see [2, 3, 4, 5, 6, 15, 16, 19, 21]. Most modern methods employ transformations with symplectic matrices to compute the desired Lagrangian invariant subspaces and to solve the algebraic Riccati equation.

Definition 3 *A $2n \times 2n$ complex matrix is called symplectic if $\mathcal{S}^H J_n \mathcal{S} = J_n$.*

For a given Hamiltonian matrix $\mathcal{H}$ the existence of a Lagrangian invariant subspace is equivalent to the existence of a symplectic similarity transformation of the Hamiltonian matrix to a Hamiltonian block triangular form, see, e.g. [19]. If there exists a symplectic matrix $\mathcal{S}$ such that

$$\mathcal{R} := \mathcal{S}^{-1}\mathcal{H}\mathcal{S} = \begin{bmatrix} R & K \\ 0 & -R^H \end{bmatrix}, \tag{4}$$

then the first n columns of $\mathcal{S}$ span a Lagrangian invariant subspace of $\mathcal{H}$ corresponding to the eigenvalues of R. The form (4) is called *Hamiltonian block triangular form* and if furthermore R is upper triangular or quasi upper triangular, then it is called *Hamiltonian triangular form.* Note that for the existence of Lagrangian invariant subspaces it is not necessary that R is triangular if one is not interested in the actual eigenvalues. Most numerical methods, however, will return a Hamiltonian triangular form.

The form (4) and necessary and sufficient conditions for the existence of symplectic similarity transformations to the form (4) in the case that $\mathcal{H}$ has no purely imaginary eigenvalues were first introduced in [20]. Necessary and sufficient conditions in the general case have first been proposed in [17], a complete proof has recently been given in terms of canonical forms for Hamiltonian matrices under symplectic similarity transformations in [18].

As we have already mentioned, the main difficulty in the analysis of canonical forms, existence of Lagrangian invariant subspaces and solutions to the algebraic Riccati equations comes from eigenvalues of the Hamiltonian matrix on the imaginary axis and the Jordan structures associated with these eigenvalues. For the analysis of the Lagrangian invariant subspaces associated with eigenvalues on the imaginary axis it is essential to know the Jordan structure under symplectic transformations, since only in this way we obtain Lagrangian invariant subspaces. In particular in the context of numerical algorithms this is even more important, since finite arithmetic computations with nonstructured transformations destroy the structure, so that Lagrangian invariant subspaces may not exist anymore, see [23] for an example. Fortunately the normal form results of [18] also immediately give the existence of similar forms under unitary symplectic transformations, which is what is needed for the purpose of developing numerically stable algorithms [2, 12].

In this paper we study existence and uniqueness of Lagrangian subspaces associated with different selections of n eigenvalues for the general problem, i.e., we allow that the Hamiltonian matrix has purely imaginary eigenvalues.

We denote by $\Lambda(A)$ the spectrum of a square matrix A, counting multiplicities. For a Hamiltonian matrix if $\lambda \in \Lambda(\mathcal{H})$ and $\operatorname{Re}\lambda \neq 0$ then $-\bar{\lambda} \in \Lambda(\mathcal{H})$, i.e., the spectrum of a Hamiltonian matrix $\mathcal{H}$ is symmetric with respect to the imaginary axis. Furthermore, if $\mathcal{H}$ has block triangular form (4) and if $i\alpha$ is a purely imaginary eigenvalue (including zero), then it must have even algebraic multiplicity. It follows that the spectrum of a Hamiltonian matrix $\mathcal{H}$ in the form (4) can be partitioned into two disjoint subsets

$$\Lambda_1(\mathcal{H}) \quad = \quad \{\underbrace{\lambda_1, \ldots, \lambda_1}_{r_1}, \underbrace{-\bar{\lambda}_1, \ldots, -\bar{\lambda}_1}_{r_1}, \ldots, \underbrace{\lambda_p, \ldots, \lambda_p}_{r_p}, \underbrace{-\bar{\lambda}_p, \ldots, -\bar{\lambda}_p}_{r_p}\}$$

$$\Lambda_2 = \{\underbrace{i\alpha_1, \ldots, i\alpha_1}_{2s_1}, \ldots, \underbrace{i\alpha_q, \ldots, i\alpha_q}_{2s_q}\},$$

where $\lambda_1, \ldots, \lambda_p$ are pairwise disjoint eigenvalues with nonzero real part and $i\alpha_1, \ldots, i\alpha_q$ are pairwise disjoint purely imaginary eigenvalues.

To obtain Lagrangian invariant subspaces we have to select the n eigenvalues associated with this subspace in a particular way. In most applications it is desired to determine Lagrangian invariant subspaces associated with eigenvalue selections of n eigenvalues

$$\omega = \{\underbrace{\lambda_1, \ldots, \lambda_1}_{r_1}, \ldots \underbrace{\lambda_p, \ldots, \lambda_p}_{r_p}, \underbrace{i\alpha_1, \ldots, i\alpha_1}_{s_1}, \ldots, \underbrace{i\alpha_q, \ldots, i\alpha_q}_{s_q}, \}$$

where of each pair $\lambda_j, -\bar{\lambda}_j$ only one of the two eigenvalues can be chosen and of the purely imaginary eigenvalues half of each of the pairwise different ones. We denote the set of all possible such selections by $\Omega(\mathcal{H})$. Observe that for each such selection ω the set $\Lambda(\mathcal{H})/\omega$ is obtained by reflecting ω at the imaginary axis. Note that $\Omega(\mathcal{H})$ contains 2^p different elements and in all elements the purely imaginary eigenvalues are the same. Note further that if $\mathcal{H}$ cannot be transformed to the block triangular form (4), then the set $\Omega(\mathcal{H})$ may not be well defined. A simple example is the matrix J_1. The complete analysis, existence, uniqueness as well as parametrizations of the different subspaces for this case are given in Section 3.1.

In the case of multiple eigenvalues with nonzero real part we can also study Lagrangian invariant subspaces that include pairs of eigenvectors associated with pairs $\lambda_j, -\bar{\lambda}_j$ which is not allowed in the set $\Omega(\mathcal{H})$. We study this general case in subsection 3.2 and show that in general we will not obtain unique Lagrangian invariant subspaces in this case except when each such eigenvalue has only one single Jordan block. We will parametrize the spaces if they are not unique.

Finally we analyse the condition for the existence of the Lagrangian invariant subspaces when the sub-blocks of the Hamiltonian matrix have additional properties. As an application we then study existence and uniqueness of Hermitian solutions to the algebraic Riccati equation in some special cases. The general analysis is still an open problem.

2 Hamiltonian triangular and block triangular forms

In this section we review some results of [18] that we need in the following analysis and we also introduce the structure inertia index associated with purely imaginary eigenvalues. We denote in the following a single Jordan block associated with an eigenvalue λ by $N_s(\lambda) = \lambda I_s + N_s$ with N_s a nilpotent Jordan block of size s. We also frequently use the antidiagonal matrices

$$P_r = \begin{bmatrix} & & -1 \\ & (-1)^2 & \\ & \cdot & \\ (-1)^r & & \end{bmatrix} \tag{5}$$

and we denote by e_j the j-th unit vector of appropriate size.

Lemma 4 *Suppose that $i\alpha$ is a purely imaginary eigenvalue of a Hamiltonian matrix $\mathcal{H}$ and the Jordan structure of the Jordan blocks associated with this eigenvalue is $N(i\alpha) := i\alpha I + N$, where*

$$N = \mathrm{diag}(N_{r_1}, \ldots, N_{r_s}). \tag{6}$$

Then there exists a full column rank matrix U such that $\mathcal{H}U = UN(i\alpha)$ and

$$U^H J_n U = \mathrm{diag}(\pi_1 P_{r_1}, \ldots, \pi_s P_{r_s}),$$

where $\pi_k \in \{1, -1\}$ if r_k is even and $\pi_k \in \{i, -i\}$ if r_k is odd.

Proof. See [18]. □

Using the indices and matrices introduced in Lemma 4, the *structure inertia index* associated with the eigenvalue $i\alpha$ is defined as

$$\mathrm{Ind}_S(i\alpha) = \{\beta_1, \ldots, \beta_s\},$$

where $\beta_k = (-1)^{\frac{r_k}{2}}\pi_k$ if r_k is even, and $\beta_k = (-1)^{\frac{r_k-1}{2}} i\pi_k$ is r_k is odd. Note that the β_i are all ± 1 and there is one index associated with each Jordan block. The structure inertia index is closely related to the sign characteristics for Hermitian pencils, see [14, 18]

The tuple $\mathrm{Ind}_S(i\alpha)$ is usually partitioned into three parts $\mathrm{Ind}_S^e(i\alpha)$, $\mathrm{Ind}_S^c(i\alpha)$, $\mathrm{Ind}_S^d(i\alpha)$, where $\mathrm{Ind}_S^e(i\alpha)$ contains all the structure inertia indices corresponding to even r_k, $\mathrm{Ind}_S^e(i\alpha)$ contains the maximal number of structure inertia indices corresponding to odd r_k in ± 1 pairs and $\mathrm{Ind}_S^d(i\alpha)$ contains the remaining indices, i.e., all indices in $\mathrm{Ind}_S^d(i\alpha)$ have the same sign. For details see [18].

Necessary and sufficient conditions for the existence of a transformation to a Hamiltonian triangular form (4) and hence also for the existence of Lagrangian invariant subspaces are given in the following Theorem.

Theorem 5 *[18] Let $\mathcal{H}$ be a Hamiltonian matrix, let $i\alpha_1, \ldots, i\alpha_\nu$ be its pairwise disjoint purely imaginary eigenvalues and let the columns of U_k, $k = 1, \ldots, \nu$, span the associated invariant subspaces of dimension t_k. Then the following are equivalent.*

i) There exists a symplectic matrix $\mathcal{S}$, such that $\mathcal{S}^{-1}\mathcal{H}\mathcal{S}$ is Hamiltonian triangular.

ii) There exists a unitary symplectic matrix $\mathcal{U}$, such that $\mathcal{U}^H\mathcal{H}\mathcal{U}$ is Hamiltonian triangular.

iii) $U_k^H J U_k$ is congruent to J_{t_k} for all $k = 1, \ldots, \nu$.

iv) $\mathrm{Ind}_S^d(i\alpha_k)$ is void for all $k = 1, \ldots, \nu$.

Moreover, if any of the equivalent condition holds, then the symplectic matrix $\mathcal{S}$ can be chosen such that $\mathcal{S}^{-1}\mathcal{H}\mathcal{S}$ is in Hamiltonian triangular Jordan form

$$\begin{bmatrix} R_r & 0 & 0 & 0 & 0 & 0 \\ 0 & R_e & 0 & 0 & D_e & 0 \\ 0 & 0 & R_c & 0 & 0 & D_c \\ 0 & 0 & 0 & -R_r^H & 0 & 0 \\ 0 & 0 & 0 & 0 & -R_e^H & 0 \\ 0 & 0 & 0 & 0 & 0 & -R_c^H \end{bmatrix}, \tag{7}$$

where the blocks with subscript r are associated with eigenvalues of nonzero real part and have the substructure

$$R_r \;=\; \operatorname{diag}(R_1^r,\ldots,R_\mu^r), \quad R_k^r = \operatorname{diag}(N_{d_{k,1}}(\lambda_k),\ldots,N_{d_{k,p_k}}(\lambda_k)), \quad k=1,\ldots,\mu.$$

The blocks with subscript e are associated with the structure inertia indices of even r_k for all purely imaginary eigenvalues and have the substructure

$$\begin{aligned} R_e &= \operatorname{diag}(R_1^e,\ldots,R_\nu^e), \quad R_k^e = \operatorname{diag}(N_{l_{k,1}}(i\alpha_k),\ldots,N_{l_{k,q_k}}(i\alpha_k)), \\ D_e &= \operatorname{diag}(D_1^e,\ldots,D_\nu^e), \quad D_k^e = \operatorname{diag}(\beta_{k,1}^e e_{l_{k,1}} e_{l_{k,1}}^H,\ldots,\beta_{k,q_k}^e e_{l_{k,q_k}} e_{l_{k,q_k}}^H). \end{aligned}$$

The blocks with subscript c are associated with paired blocks of inertia indices associated with odd-sized blocks for all purely imaginary eigenvalues and have the substructure

$$\begin{aligned} R_c &= \operatorname{diag}(R_1^c,\ldots,R_\nu^c), \quad R_k^c = \operatorname{diag}(B_{k,1},\ldots,B_{k,r_k}), \\ D_c &= \operatorname{diag}(D_1^c,\ldots,D_\nu^c), \quad D_k^c = \operatorname{diag}(C_{k,1},\ldots,C_{k,r_k}), \end{aligned}$$

where

$$\begin{aligned} B_{k,j} &= \begin{bmatrix} N_{m_{k,j}}(i\alpha_k) & 0 & -\frac{\sqrt{2}}{2}e_{m_{k,j}} \\ 0 & N_{n_{k,j}}(i\alpha_k) & -\frac{\sqrt{2}}{2}e_{n_{k,j}} \\ 0 & 0 & i\alpha_k \end{bmatrix}, \\ C_{k,j} &= \frac{\sqrt{2}}{2} i\beta_{k,j}^c \begin{bmatrix} 0 & 0 & e_{m_{k,j}} \\ 0 & 0 & -e_{n_{k,j}} \\ -e_{m_{k,j}}^H & e_{n_{k,j}}^H & 0 \end{bmatrix}. \end{aligned}$$

Since the existence of a symplectic similarity transformation to Hamiltonian block triangular form is equivalent to the existence of Lagrangian invariant subspaces, we require in the following that the Hamiltonian matrix has a Hamiltonian triangular form.

To give the complete analysis for the case of purely imaginary eigenvalues we need the following technical lemma which was essentially given in [18].

Lemma 6 *Given pairs of matrices $(\pi_k P_{r_k}, N_{r_k})$, $k=1,2$, where r_1, r_2 are either both even or both odd. Let $\pi_1,\pi_2 \in \{1,-1\}$ if both r_k are even and $\pi_1,\pi_2 \in \{i,-i\}$ if both r_k are odd, let*

$$(\mathcal{P}_c,\mathcal{N}_c) := \left(\begin{bmatrix} \pi_1 P_{r_1} & 0 \\ 0 & \pi_2 P_{r_2} \end{bmatrix}, \begin{bmatrix} N_{r_1} & 0 \\ 0 & N_{r_2} \end{bmatrix} \right)$$

and let $d := \frac{|r_1-r_2|}{2}$. *If* $\pi_1 = (-1)^{d+1}\pi_2$, *i.e.,* $\beta_1 = -\beta_2$ *for the corresponding* β_1 *and* β_2, *then we have the following transformations to Hamiltonian triangular form.*

1. *If* $r_1 \geq r_2$ *then with*

$$Z_1 := \begin{bmatrix} I_d & 0 & 0 & 0 \\ 0 & I_{r_2} & 0 & -\frac{1}{2}\bar{\pi}_2 P_{r_2}^{-1} \\ 0 & 0 & \bar{\pi}_1 P_d^{-1} & 0 \\ 0 & -I_{r_2} & 0 & -\frac{1}{2}\bar{\pi}_2 P_{r_2}^{-1} \end{bmatrix}$$

we obtain $Z_1^H \mathcal{P}_c Z_1 = J_{\frac{r_1+r_2}{2}}$ *and*

$$Z_1^{-1}\mathcal{N}_c Z_1 = \begin{bmatrix} N_{\frac{r_1+r_2}{2}} & K \\ 0 & -N_{\frac{r_1+r_2}{2}}^H \end{bmatrix},$$

where $K = -\frac{1}{2}(\pi_2 e_d e_{\frac{r_1+r_2}{2}}^H + \bar{\pi}_2 e_{\frac{r_1+r_2}{2}} e_d^H)$.

2. *If* $r_1 < r_2$, *then with*

$$Z_2 = \begin{bmatrix} \pi_1 P_{r_1} & 0 & \frac{1}{2} I_{r_1} & 0 \\ 0 & \pi_2 P_d & 0 & 0 \\ -\pi_1 P_{r_1} & 0 & \frac{1}{2} I_{r_1} & 0 \\ 0 & 0 & 0 & I_d \end{bmatrix}$$

we obtain that $Z_2^H \mathcal{P}_c Z_2 = J_{\frac{r_1+r_2}{2}}$ *and*

$$Z_2^{-1}\mathcal{N}_c Z_2 = \begin{bmatrix} -N_{\frac{r_1+r_2}{2}}^H & K \\ 0 & N_{\frac{r_1+r_2}{2}}^H \end{bmatrix},$$

where $K = -\frac{1}{2}(\pi_1 e_1 e_{r_1+1}^H + \bar{\pi}_1 e_{r_1+1} e_1^H)$.

Proof. The proof is a simple modification of the proof of Lemma 18 in [18]. ☐

Finally we recall a basic result on the Hermitian solutions of Lyapunov equations which follows directly from the general characterization of the solutions, see [7, 13].

Lemma 7 *Let* $W = W^H = [w_{ij}] \in \mathbf{C}^{n\times n}$. *Then* $X \in \mathbf{C}^{n\times n}$ *is an Hermitian solution of the (singular) Lyapunov equation*

$$N_n X + X N_n^T = W \tag{8}$$

if and only for $k = 1, \ldots, n$, *we have*

$$(-1)^{\frac{n+k}{2}} w_{\frac{n+k}{2},\frac{n+k}{2}} + 2\,\mathrm{Re}\Big(\sum_{i=k}^{\frac{n+k}{2}-1} (-1)^i w_{i,n+k-i}\Big) = 0$$

if $n+k$ *is even, and*

$$\mathrm{Im}\Big(\sum_{i=k}^{\frac{n+k-1}{2}} (-1)^i w_{i,n+k-i}\Big) = 0$$

if $n+k$ *is odd.*

Proof. The proof follows directly by backward substitution in (8). ☐

3 Lagrangian invariant subspaces

In this section we discuss existence and uniqueness of Lagrangian invariant subspaces for Hamiltonian matrices. We split the analysis into two cases. In the first subsection 3.1 we restrict the eigenvalue selection to the set $\Omega(\mathcal{H})$ and in subsection 3.2 we study the general case.

3.1 Eigenvalue selections in $\Omega(\mathcal{H})$

In this subsection we study eigenvalue selections in $\Omega(\mathcal{H})$ as defined in Section 1. Note that in this paper we only consider complex problems. In the real case we obtain the same results if we include the restriction that all complex conjugate pairs must be contained in the same ω. The first lemma shows that for every eigenvalue selection $\omega \in \Omega(\mathcal{H})$ we obtain an associated Lagrangian invariant subspace.

Lemma 8 *Let $\mathcal{H}$ have a Hamiltonian block triangular form and let $\omega \in \Omega(\mathcal{H})$. Then there exists a (unitary) symplectic matrix $\mathcal{S}$ such that*

$$\mathcal{S}^{-1}\mathcal{H}\mathcal{S} = \begin{bmatrix} R & K \\ 0 & -R^H \end{bmatrix},$$

where $\Lambda(R) = \omega$ and where the first n columns of $\mathcal{S}$ span the associated Lagrangian invariant subspace.

Proof. It is easy to determine from (7) a basis for the invariant subspace corresponding to each purely imaginary eigenvalue. For the remaining eigenvalues if $\lambda_k \notin \omega$ we can exchange the corresponding blocks in $\begin{bmatrix} R_k^r & 0 \\ 0 & -(R_k^r)^H \end{bmatrix}$ by performing a symplectic similarity transformation with J of appropriate size. In this way we can reorder the diagonal blocks, such that $\Lambda(R) = \omega$. $\square$

By Lemma 8 for $\omega \in \Omega(\mathcal{H})$, the related Lagrangian invariant subspaces are spanned by all chains of root vectors corresponding to $\lambda \in \omega$ with $\operatorname{Re}\lambda \neq 0$ and parts of the chains of the root vectors corresponding to $i\alpha \in \omega$. Hence the uniqueness of the Lagrangian invariant subspace is completely determined by the Jordan structure of the purely imaginary eigenvalues.

Furthermore, using the Hamiltonian triangular form (7) it is sufficient to study the uniqueness of Lagrangian invariant subspace for each small Hamiltonian submatrix

$$\begin{bmatrix} R_k^e & 0 & D_k^e & 0 \\ 0 & R_k^c & 0 & D_k^c \\ 0 & 0 & -(R_k^e)^H & 0 \\ 0 & 0 & 0 & -(R_k^c)^H \end{bmatrix},$$

associated with a single purely imaginary eigenvalue $i\alpha_k$.

Lemma 9 *Let $\mathcal{H}$ be a Hamiltonian matrix with only two Jordan blocks $N_{r_1}(i\alpha), N_{r_2}(i\alpha)$. Suppose that the corresponding structure inertia indices satisfy $\beta_1 = -\beta_2$. Then there are infinitely many Lagrangian invariant subspaces of $\mathcal{H}$ which can be parametrized via the solution set of a homogeneous Lyapunov equation.*

Proof. We may assume without loss of generality that $i\alpha = 0$, i.e., $N_{r_1}(i\alpha) = N_{r_1}$ and $N_{r_2}(i\alpha) = N_{r_2}$ are nilpotent. If this is not the case then the nilpotent Hamiltonian matrix $\mathcal{H} - i\alpha I$ has this property, and clearly the invariant subspaces are the same as those of $\mathcal{H}$.

By Lemma 4 there exists a matrix $\mathcal{Z}$ such that

$$\mathcal{Z}^{-1}\mathcal{H}\mathcal{Z} = \mathrm{diag}(N_{r_1}, N_{r_2}), \quad \mathcal{Z}^H J \mathcal{Z} = \mathrm{diag}(\pi_1 P_{r_1}, \pi_2 P_{r_2})$$

and, since the size of $\mathcal{H}$ is even, r_1 and r_2 must be both even or both odd. Since $\beta_1 = -\beta_2$, by Lemma 6 the Hamiltonian block triangular form always exists and is the same, regardless whether r_1 and r_2 are both odd or both even. Hence, without loss of generality we may consider the even case and set $r_1 = 2m_1$ and $r_2 = 2m_2$. Let us assume that $m_1 \geq m_2$. (If $m_1 < m_2$, then just interchange N_{r_1} and N_{r_2} as well as $\pi_1 P_{r_1}$ and $\pi_2 P_{r_2}$.) Then $d = m_1 - m_2$ and we set $m = m_1 + m_2$. Following Lemma 6 we obtain

$$\mathcal{N} := \mathcal{S}^{-1}\mathcal{H}\mathcal{S} = \begin{bmatrix} N_m & K \\ 0 & -N_m^H \end{bmatrix}, \tag{9}$$

where $K = (\tau e_d e_m^H + \bar{\tau} e_m e_d^H)$, and $\tau = -\frac{1}{2}\pi_2$.

We will determine all possible symplectic matrices that transform $\mathcal{H}$ to Hamiltonian block triangular form (4). Since we have already obtained (9) this is equivalent to determining all symplectic matrices $\mathcal{U}$ such that $\mathcal{U}^{-1}\mathcal{N}\mathcal{U}$ stays Hamiltonian block triangular. Since every symplectic matrix has an SR decomposition with a unitary symplectic S and symplectic block triangular R, see [3], it suffices to consider the case that $\mathcal{U}$ is unitary symplectic. Suppose that

$$\mathcal{N}\mathcal{U} = \mathcal{U} \begin{bmatrix} A & D \\ 0 & -A^H \end{bmatrix} =: \mathcal{U}\mathcal{A}. \tag{10}$$

Every unitary symplectic matrix has the form $\mathcal{U} = \begin{bmatrix} U_1 & U_2 \\ -U_2 & U_1 \end{bmatrix}$ with $n \times n$ matrices U_1, U_2, see [20] and we can find a block diagonal unitary symplectic matrix $\mathrm{diag}(Q, Q)$, such that

$$\mathcal{U} \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} = \left[\begin{array}{cc|cc} U_{11} & U_{12} & 0 & U_{22} \\ 0 & -U_{22} & U_{11} & U_{12} \end{array}\right],$$

with U_{22} of full column rank q. Clearly $\operatorname{rank} U_{11} = m - q =: p$. Partitioning the block A in $\mathcal{A}$ conformally as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

and comparing the $(2,1)$ blocks on both sides of (10) we obtain

$$N_m^T \begin{bmatrix} 0 & U_{22} \end{bmatrix} + \begin{bmatrix} 0 & U_{22} \end{bmatrix} A = 0,$$

which implies that $A_{21} = 0$ and

$$N_m^T U_{22} + U_{22} A_{22} = 0. \tag{11}$$

Comparison of the $(1,1)$ blocks yields $N_m U_1 - K U_2 = U_1 A$ and from the first p columns we get

$$N_m U_{11} = U_{11} A_{11}. \tag{12}$$

Since U_{11} and U_{22} are of full column rank, (11) and (12) imply that both A_{11} and A_{22} are nilpotent. Moreover, since $\operatorname{rank} N_m = m - 1$, there must exist matrices Z_1 and Z_2 such that

$$Z_1^{-1} A_{11} Z_1 = N_p, \qquad Z_2^{-1} A_{22} Z_2 = N_q.$$

Let $V_{11} = U_{11} Z_1$ and $V_{22} = U_{22} Z_2$, then

$$N_m V_{11} = V_{11} N_p, \qquad N_m^T V_{22} + V_{22} N_q = 0.$$

Using the simple fact that $P_r^H N_r^H P_r = -N_r$ and Lemma 4.4.11 in [10], we obtain that

$$V_{11} = \begin{bmatrix} T_1 \\ 0 \end{bmatrix}, \qquad V_{22} = \begin{bmatrix} 0 \\ P_q T_2 \end{bmatrix},$$

where T_1 and T_2 are $p \times p$ and $q \times q$ nonsingular upper triangular Toeplitz matrices, respectively. Set $\hat{\mathcal{U}} = \mathcal{U}\mathcal{X}_1$, where

$$\mathcal{X}_1 = \operatorname{diag}(Z_1 T_1^{-1}, Z_2 T_2^{-1} P_q^{-1}, (Z_1 T_1^{-1})^{-H}, (Z_2 T_2^{-1} P_q^{-1})^{-H}),$$

then $\hat{\mathcal{U}}$ is symplectic (but not unitary any more), and if we partition $\hat{\mathcal{U}} = [\hat{U}_1, \hat{U}_2]$ with $\hat{U}_1, \hat{U}_2 \in \mathbf{C}^{2m \times m}$, then

$$\hat{U}_1 = \begin{bmatrix} I_p & \hat{U}_{12} \\ 0 & \hat{U}_{22} \\ 0 & 0 \\ 0 & -I_q \end{bmatrix}.$$

Set $Z_3 = \begin{bmatrix} I_p & -\hat{U}_{12} \\ 0 & I_q \end{bmatrix}$ and let $\mathcal{X}_2 = \operatorname{diag}(Z_3, Z_3^{-H})$. Then $\tilde{\mathcal{U}} := \hat{\mathcal{U}}\mathcal{X}_2$ is still symplectic and has the first m columns

$$\tilde{U}_1 = \hat{U}_1 Z_3 = \begin{bmatrix} I_p & 0 \\ 0 & W \\ 0 & 0 \\ 0 & -I_q \end{bmatrix}.$$

Moreover, since $\tilde{\mathcal{U}}$ is symplectic, we have $W = W^H$ and we obtain that

$$\mathcal{V} = \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & W & 0 & I_q \\ 0 & 0 & I_p & 0 \\ 0 & -I_q & 0 & 0 \end{bmatrix}$$

is symplectic. Since $\mathcal{V}$ and $\tilde{\mathcal{U}}$ are both symplectic and have the same first m columns, there must exist a symplectic upper triangular matrix $\mathcal{X}_3 = \begin{bmatrix} I_m & X \\ 0 & I_m \end{bmatrix}$ such that $\tilde{\mathcal{U}} = \mathcal{V}\mathcal{X}_3$. Then $\mathcal{X} := \mathcal{X}_3(\mathcal{X}_1\mathcal{X}_2)^{-1}$ is symplectic block triangular and $\mathcal{U} = \mathcal{V}\mathcal{X}$. Note that

$$\mathcal{V}^{-1}\mathcal{N}\mathcal{V} = \mathcal{X}\mathcal{A}\mathcal{X}^{-1} =: \hat{\mathcal{A}}.$$

It remains to study the existence of an Hermitian matrix W such that $\hat{\mathcal{A}}$ is Hamiltonian block triangular. Using the block form of $\mathcal{N}$ in (9), the Hermitian matrix W must satisfy

$$N_q W + W N_q^H = K_{22}, \tag{13}$$

where $K_{22} = 0$ if $d \leq p$ $(q \leq r_2)$ or $K_{22} = \tau e_{d-p} e_q^H + \bar{\tau} e_q e_{d-p}^H$ if $d > p$. In the first case the singular Lyapunov equation (13) has infinitely many Hermitian solutions W, see [7, 10]. In the second case, since we have assumed that r_1, r_2 are both even, it follows that $\tau \neq 0$ is real. Since $d-p+q = 2(q-m_2)$ is even and the sum of the $(d-p)$th lower anti-diagonal elements of $P_q K_{22}$ is $(-1)^{d-p}2\tau$, which is nonzero, it follows by Lemma 7 that equation (13) has no solution. The case that r_1, r_2 are both odd can be analysed in the same way.

Finally, combining this analysis with (9) we get that all symplectic matrices that leave the Hamiltonian block triangular form invariant are of the form

$$(\mathcal{S} \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & W & 0 & I_q \\ 0 & 0 & I_p & 0 \\ 0 & -I_q & 0 & 0 \end{bmatrix})\mathcal{X},$$

with $q \leq r_2$, $W = W^H$ satifying (13) and $\mathcal{X}$ symplectic block triangular. Since $\mathcal{X}$ does not effect the Lagrangian invariant subspaces, we obtain that all Lagrangian invariant subspaces of $\mathcal{H}$ can be parametrized via W as

$$\mathcal{S} \begin{bmatrix} I_p & 0 \\ 0 & W \\ 0 & 0 \\ 0 & -I_q \end{bmatrix}, \tag{14}$$

where $W = W^H$, $q \leq r_2$ and W satisfies $N_q W + W N_q^H = 0$. $\square$

Note that the construction in the proof of Lemma 9 yields a parametrization of the infinite number of solutions as a linear manifold, which describes the solution set of the homogeneous Lyapunov equation (13). It follows that the real dimension of the solution manifold is q, which is the rank of the block U_2 in the transformation matrix $\mathcal{U}$ in (10).

Another immediate consequence of Lemma 9 is that for a nilpotent Hamiltonian matrix $\mathcal{H}$, if Ind_S^c is not void or if Ind_S^e contains at least two different structure inertia indices of opposite sign, then $\mathcal{H}$ has infinitely many Lagrangian invariant subspaces. A general parametrization of all the Lagrangian invariant subspaces can be derived in a similar way as in the case of two Jordan blocks.

We have seen that for a Hamiltonian matrix with a single purely imaginary eigenvalue we obtain a necessary condition for the uniqueness of the Lagrangian invariant subspace. The next lemma shows that this condition is also sufficient.

Lemma 10 *Let $\mathcal{H}$ be Hamiltonian and have only a single purely imaginary eigenvalue $i\alpha$. Then the following statements are equivalent.*

i) *There exists a unique Lagrangian invariant subspace of $\mathcal{H}$.*

ii) *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are symplectic matrices such that both $\mathcal{S}_1^{-1}\mathcal{H}\mathcal{S}_1$, $\mathcal{S}_2^{-1}\mathcal{H}\mathcal{S}_2$ are Hamiltonian block triangular, then $\mathcal{S}_1^{-1}\mathcal{S}_2$ is symplectic block triangular.*

iii) *$\mathcal{H}$ has only even size Jordan blocks with structure inertia indices of the same sign, i.e., there exists a symplectic matrix $\mathcal{S}$, such that*

$$\mathcal{S}^{-1}\mathcal{H}\mathcal{S} = \begin{bmatrix} R & K \\ 0 & -R^H \end{bmatrix} =: \mathcal{R}, \tag{15}$$

with $R = \mathrm{diag}(N_{l_1}, \ldots, N_{l_q})$, $K = \beta\,\mathrm{diag}(e_{l_1}e_{l_1}^H, \ldots, e_{l_q}e_{l_q}^H)$, and $\beta = 1$ or $\beta = -1$.

iv) *For every Hamiltonian block triangular form $\hat{\mathcal{R}} = \begin{bmatrix} A & D \\ 0 & -A^H \end{bmatrix}$ of $\mathcal{H}$, if the columns of Φ form a basis of the left eigenvector subspace of A, i.e., $\Phi^H A = i\alpha\Phi^H$, then $\Phi^H D\Phi$ is positive definite or negative definite.*

Proof. As in the proof of Lemma 9, we may assume without loss of generality that $\mathcal{H}$ is nilpotent. It follows that the matrix whose columns span the invariant subspace to the eigenvalue 0 is a nonsingular matrix X and since X^HJX is congruent to J, by Theorem 5 $\mathcal{H}$ always has a Hamiltonian triangular form.

i) $\Leftrightarrow$ **ii)** is obvious.

ii) $\Rightarrow$ **iii)** Suppose that iii) does not hold. Then there exists at least one pair of structure inertia indices of opposite sign in Ind_S^e or Ind_S^c. Without loss of generality we may assume that $\mathcal{H}$ has only two Jordan blocks. Otherwise the construction can be repeated. Let $\mathcal{S}$ be a symplectic matrix such that $\mathcal{S}^{-1}\mathcal{H}\mathcal{S}$ is Hamiltonian triangular. By Lemma 9 we can determine two symplectic matrices

$$\mathcal{S}_1 = \mathcal{S}\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & W_1 & 0 & I \\ 0 & 0 & I & 0 \\ 0 & -I & 0 & 0 \end{bmatrix}\hat{\mathcal{X}}_1, \quad \mathcal{S}_2 = \mathcal{S}\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & W_2 & 0 & I \\ 0 & 0 & I & 0 \\ 0 & -I & 0 & 0 \end{bmatrix}\hat{\mathcal{X}}_2,$$

with $W_k = W_k^T$, $\hat{\mathcal{X}}_k$ symplectic block triangular and $W_1 \neq W_2$, such that $\mathcal{S}_1^{-1}\mathcal{H}\mathcal{S}_1$ and $\mathcal{S}_2^{-1}\mathcal{H}\mathcal{S}_2$ are Hamiltonian block triangular. Using the expressions for $\mathcal{S}_1$, $\mathcal{S}_2$, we have

$$\mathcal{S}_1^{-1}\mathcal{S}_2 = \hat{\mathcal{X}}_1^{-1} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & W_2 - W_1 & 0 & I \end{bmatrix} \hat{\mathcal{X}}_2.$$

Since $W_1 \neq W_2$, it follows that $\mathcal{S}_1^{-1}\mathcal{S}_2$ cannot be symplectic block triangular, which contradicts **ii)**.

iii) $\Rightarrow$ **ii)** Equivalently we may show that a symplectic matrix $\mathcal{Z}$, such that $\mathcal{Z}^{-1}\mathcal{R}\mathcal{Z}$ is Hamiltonian block triangular, must be symplectic block triangular. Here $\mathcal{R}$ is defined as in (15). Set $\mathcal{Z}^{-1}\mathcal{R}\mathcal{Z} =: \hat{\mathcal{H}} := \begin{bmatrix} A & D \\ 0 & -A^H \end{bmatrix}$ and partition $\mathcal{Z} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$. Then

$$RZ_{11} + KZ_{21} = Z_{11}A \tag{16}$$

and

$$-R^H Z_{21} = Z_{21}A. \tag{17}$$

If $Z_{21} \neq 0$, by (17) it follows that range Z_{21} is an invariant subspace of $-R^H$. Hence, there exists a vector x such that $Z_{21}x \neq 0$ and

$$R^H Z_{21}x = 0, \tag{18}$$

i.e., $Z_{21}x$ is left eigenvector of R. Multiplying $(Z_{21}x)^H$ and x on both sides of (16) and using (18) we get

$$(Z_{21}x)^H K(Z_{21}x) = -x^H Z_{21}^H Z_{11}Ax.$$

Since $\mathcal{Z}$ is symplectic, we have $Z_{21}^H Z_{11} = Z_{11}^H Z_{21}$. Combining (17) and (18) we get

$$x^H Z_{21}^H Z_{11}Ax = x^H Z_{11}^H Z_{21}Ax = -x^H Z_{11}^H R^H Z_{21}x = 0$$

and, therefore,

$$(Z_{21}x)^H K(Z_{21}x) = 0.$$

On the other hand, since $Z_{21}x$ is a left eigenvector of R, by the structure of R there must exist a nonzero vector y such that $Z_{21}x = Ey$, where

$$E := [e_{m_1}, \dots, e_{m_q}], \tag{19}$$

with $m_k = \sum_{p=1}^{k} l_p$ for $k = 1, \dots, q$. But $E^H KE = \beta I_q$ and hence

$$0 = (Z_{21}x)^H K(Z_{21}x) = y^H E^H KEy = \beta y^H y \neq 0,$$

which is a contradiction.

iii) $\Rightarrow$ **iv)** Since both $\mathcal{R}$ and $\hat{\mathcal{R}}$ are Hamiltonian block triangular, by **ii)** there exists a symplectic block triangular matrix $\mathcal{S} = \begin{bmatrix} S_1 & S_2 \\ 0 & S_1^{-H} \end{bmatrix}$ such that $\hat{\mathcal{R}} = \mathcal{S}^{-1}\mathcal{R}\mathcal{S}$. Hence $S_1^{-1}RS_1 = A$ and $D = S_1^{-1}RS_2 + S_1^{-1}KS_1^{-H} + S_2^H R^H S_1^{-H}$. Since A is similar to R we can take $\Phi = S_1^H E$ with E as in (19). Then it follows by a simple calculation that $\Phi^H D\Phi = \beta I_q$.

iv) $\Rightarrow$ **iii)** Let the canonical form of $\mathcal{H}$ as in Theorem 5 be $\mathcal{R} = \begin{bmatrix} R & K \\ 0 & -R^H \end{bmatrix}$. If there exists (at least) one pair of structure inertia indices of opposite sign in Ind_S^e or Ind_S^c, from the canonical form, it is easy to construct a vector x such that $x^H R = 0$ and $x^H K x = 0$. Since $\mathcal{R}$ is also Hamiltonian triangular this contradicts **iv)**. $\square$

We summarize the results of Lemma 9 and Lemma 10 in the following Theorem.

Theorem 11 *Given a Hamiltonian matrix $\mathcal{H}$ with distinct purely imaginary eigenvalues (including the eigenvalue zero) $i\alpha_1, \ldots, i\alpha_\nu$. Suppose that $\mathcal{H}$ has a Hamiltonian block triangular form. Then for every element in $\Omega(\mathcal{H})$ there exists a unique Lagrangian invariant subspace if and only if* $\mathrm{Ind}_S^c(i\alpha_k)$ *is void and* $\mathrm{Ind}_S^e(i\alpha_k)$ *contains the same structure inertia indices for all $k = 1, \ldots, \nu$.*

Proof. Apply Lemmas 8, 9, 10. $\square$

We see that for a Hamiltonian matrix $\mathcal{H}$ with purely imaginary eigenvalues, the associated structure inertia indices play the key role for the existence of a Hamiltonian block triangular form and Lagrangian invariant subspaces. If Ind_S^d is void for all purely imaginary eigenvalues then the Hamiltonian block triangular form and the associated Lagrangian invariant subspace exist. If, furthermore, all Ind_S^c are void and no Ind_S^e contains a pair of indices of opposite sign, then for an arbitrary element $\omega \in \Omega(\mathcal{H})$ the Hamiltonian block triangular form (4) with $\Lambda(R) = \omega$ is unique up to a symplectic similarity transformation with a symplectic block triangular matrix, and the related Lagrangian invariant subspace is unique.

3.2 Multiple eigenvalues with nonzero real part

The eigenvalue restriction $\Lambda(R) \in \Omega(\mathcal{H})$ is common in many applications in particular when one is interested in stabilizing or semistabilizing solutions of Riccati equations [14, 24]. But theoretically it is also reasonable to study Lagrangian invariant subspaces associated with eigenvalue selections which include pairs of eigenvalues $\lambda, -\bar{\lambda} \in \Lambda(R)$ provided the algebraic multiplicity of λ as well as $-\bar{\lambda}$ is larger than one. It is clear that we get Lagrangian invariant subspaces also for these cases, and again we sometimes have unique subspaces and in other cases we have infinitely many solution which can be parametrized via the solutions of homogeneous matrix equations. Consider the following examples.

Example 1 The Hamiltonian matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

has a unique Lagrangian invariant subspace corresponding to $\{1, -1\}$, while for the Hamiltonian matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{20}$$

the Lagrangian invariant subspaces corresponding to $\{1, -1\}$, are

$$\text{range} \begin{bmatrix} \gamma_{11} & 0 \\ 0 & \gamma_{22} \\ 0 & \gamma_{32} \\ \gamma_{41} & 0 \end{bmatrix},$$

where $\gamma_{11}, \gamma_{22}, \gamma_{32}, \gamma_{41} \in \mathbf{C}$ satisfy $\bar{\gamma}_{11}\gamma_{32} = \bar{\gamma}_{41}\gamma_{22}$. It follows that in this case there exist infinitely many Lagrangian invariant subspaces to an eigenvalue set not in $\Omega(\mathcal{H})$. For both matrices there are also other invariant subspaces associated with eigenvalues in $\Omega(\mathcal{H})$.

Let us now carry out the analysis for eigenvalue selections which are not in $\Omega(\mathcal{H})$. Since for the Jordan structure associated with eigenvalues λ, $-\bar{\lambda}$ with nonzero real part, we can block-diagonalize the matrix and obtain a Hamiltonian Jordan form, see [18], it suffices to discuss the case that the Hamiltonian matrix $\mathcal{H}$ has only two different eigenvalues λ, $-\bar{\lambda}$. Then, see [18], there exists a symplectic matrix $\mathcal{S}$ such that

$$\mathcal{R} := \mathcal{S}^{-1}\mathcal{H}\mathcal{S} = \begin{bmatrix} \lambda I + N & 0 \\ 0 & -\bar{\lambda} I - N^H \end{bmatrix}, \tag{21}$$

where $N = \mathrm{diag}(N_{n_1}, \ldots, N_{n_l})$. Since the existence of Lagrangian invariant subspaces is equivalent to the existence of symplectic similarity transformations to Hamiltonian triangular form, the existence of a Lagrangian invariant subspace associated with eigenvalues λ, $-\bar{\lambda}$ is equivalent to the existence of a symplectic matrix $\mathcal{U}$ such that

$$\mathcal{R}\mathcal{U} = \mathcal{U} \begin{bmatrix} \lambda I_p + M_1 & 0 & 0 & L \\ 0 & -\bar{\lambda} I_q + M_2 & L^H & 0 \\ 0 & 0 & -\bar{\lambda} I_p - M_1^H & 0 \\ 0 & 0 & 0 & \lambda I_q - M_2^H \end{bmatrix} =: \mathcal{U}\tilde{\mathcal{R}},$$

where $p + q = n$ and M_1, M_2 are nilpotent. We now study the structure of $\mathcal{U}$ satisfying this relation. Let

$$\mathcal{U} := \begin{bmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ U_{21} & U_{22} & U_{23} & U_{24} \end{bmatrix}$$

be partitioned conformally with $\tilde{\mathcal{R}}$. By comparing the blocks and using the block form of $\mathcal{R}$ in (21) we get that the blocks $U_{12}, U_{13}, U_{21}, U_{24}$ vanish and, since $\mathcal{U}$ is nonsingular, U_{11} must be of full column rank. Let V be such that $V^{-1}U_{11} = \begin{bmatrix} Z_1 \\ 0 \end{bmatrix}$ with Z_1 nonsingular and let $\mathcal{V} := \mathrm{diag}(V, V^{-H})$, $\hat{\mathcal{R}} := \mathcal{V}^{-1}\mathcal{R}\mathcal{V}$ and $\hat{\mathcal{U}} := \mathcal{V}^{-1}\mathcal{U}$. Since $\mathcal{V}$ is symplectic, also $\hat{\mathcal{U}}$ is symplectic and it is easy to verify that

$$\begin{aligned}
\hat{\mathcal{U}} &= \begin{bmatrix} Z_1 & 0 & 0 & Z_2 \\ 0 & 0 & 0 & Z_3 \\ 0 & 0 & Z_1^{-H} & 0 \\ 0 & -Z_3^{-H} & -Z_3^{-H}Z_2^H Z_1^{-H} & 0 \end{bmatrix} \\
&= \begin{bmatrix} I_p & 0 & 0 & 0 \\ 0 & 0 & 0 & I_q \\ 0 & 0 & I_p & 0 \\ 0 & -I_q & 0 & 0 \end{bmatrix} \begin{bmatrix} Z_1 & 0 & 0 & Z_2 \\ 0 & Z_3^{-H} & Z_3^{-H}Z_2^H Z_1^{-H} & 0 \\ 0 & 0 & Z_1^{-H} & 0 \\ 0 & 0 & 0 & Z_3 \end{bmatrix} \\
&=: \mathcal{K}\mathcal{Z},
\end{aligned}$$

and $\mathcal{Z}$ is symplectic block triangular. Using the block structure of $\mathcal{Z}$ we then get

$$\hat{\mathcal{R}}\mathcal{K} = \mathcal{K} \begin{bmatrix} \lambda I_p + \hat{M}_1 & 0 & 0 & \hat{L} \\ 0 & -\bar{\lambda} I_q + \hat{M}_2 & \hat{L}^H & 0 \\ 0 & 0 & -\bar{\lambda} I_p - \hat{M}_1^H & 0 \\ 0 & 0 & 0 & \lambda I_q - \hat{M}_2^H \end{bmatrix}, \tag{22}$$

and

$$\hat{\mathcal{R}} = \begin{bmatrix} \lambda I_p + \hat{M}_1 & \hat{L} & 0 & 0 \\ 0 & \lambda I_q - \hat{M}_2^H & 0 & 0 \\ 0 & 0 & -\bar{\lambda} I_p - \hat{M}_1^H & 0 \\ 0 & 0 & -\hat{L}^H & -\bar{\lambda} I_q + \hat{M}_2 \end{bmatrix}. \tag{23}$$

This implies that for every p with $0 \le p \le n$, if $\mathcal{H}$ has a Hamiltonian block triangular form as $\hat{\mathcal{R}}$, then $\mathcal{H}$ always has a Lagrangian invariant subspace associated with p copies of λ and q copies of $-\bar{\lambda}$. Such a Lagrangian invariant subspace always exists, because $\mathcal{R}$ in (21) already has the form as $\hat{\mathcal{R}}$. Clearly the Lagrangian invariant subspaces are different for different p.

However, even if p is fixed, it is possible to have infinitely many Lagrangian invariant subspaces. To see this we characterize the symplectic matrices which transform $\mathcal{H}$ to $\hat{\mathcal{R}}$ in (23). We consider three cases: a) there is only one Jordan block for λ; b) there are two Jordan blocks for λ and c) there are more than two Jordan blocks for λ.

a) In this case $N = N_n$. For a given p with $0 \le p \le n$ let $\mathcal{U}$ be a symplectic matrix such that $\mathcal{R}\mathcal{U} = \mathcal{U}\hat{\mathcal{R}}$, where $\mathcal{R}$ is in (21). Since $\lambda \ne -\bar{\lambda}$, $\mathcal{U}$ must be a block diagonal matrix as $\mathrm{diag}(U, U^{-H})$ and we obtain

$$(\lambda I_n + N)U = U(\lambda I_n + \begin{bmatrix} \hat{M}_1 & \hat{L} \\ 0 & -\hat{M}_2^H \end{bmatrix}), \tag{24}$$

or equivalently

$$NU = U \begin{bmatrix} \hat{M}_1 & \hat{L} \\ 0 & -\hat{M}_2^H \end{bmatrix}.$$

Partition U as $[U_1, U_2]$, then $NU_1 = U_1 \hat{M}_1$. Since N is nilpotent, $\operatorname{rank} N = n-1$ and U_1 is of full column rank, it follows that $\hat{M}_1$ is nilpotent and $\hat{M}_1 = p-1$. Let Z_1 be nonsingular such that $Z_1^{-1} \hat{M}_1 Z_1 = N_p$. Then $N(U_1 Z_1) = (U_1 Z_1) N_p$, which implies that $U_1 Z_1 = \begin{bmatrix} T \\ 0 \end{bmatrix}$, where the $p \times p$ matrix T is a nonsingular upper triangular Toeplitz matrix, see Lemma 4.4.11 in [10]. Then it follows that U is block upper triangular and has the form $\begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}$. In this case we obtain that the related Lagrangian invariant subspace is unique.

b) If we have two Jordan blocks, say $N = \operatorname{diag}(N_r, N_{n-r})$, then we have the following characterization of the nonsingular matrix U such that

$$NU = U \begin{bmatrix} \hat{M}_1 & \hat{L} \\ 0 & -\hat{M}_2^H \end{bmatrix} =: UC.$$

Partition U as $\begin{bmatrix} U_1 & U_3 \\ U_2 & U_4 \end{bmatrix}$ with $U_1 \in \mathbf{C}^{r \times p}$, $U_4 \in \mathbf{C}^{(n-r) \times q}$. Suppose that $\operatorname{rank} U_2 = s \leq \min\{n-r, p\}$, then there exists a nonsingular matrix Y_1 such that $U_2 Y_1 = [0, U_{22}]$ with $\operatorname{rank} U_{22} = s$. With $Z_1 := \operatorname{diag}(Y_1, I)$ and $\hat{U} := U Z_1$, then $\hat{U} = \begin{bmatrix} U_{11} & U_{12} & U_3 \\ 0 & U_{22} & U_4 \end{bmatrix}$, and

$$N \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix} (Y_1^{-1} \hat{M}_1 Y_1).$$

Since $N = \operatorname{diag}(N_r, N_{n-r})$, using a similar construction as in case a), there exists a nonsingular block diagonal matrix $Y_2 = \operatorname{diag}(Y_{11}, Y_{22})$, such that

$$(Y_1 Y_2)^{-1} \hat{M}_1 (Y_1 Y_2) = \begin{bmatrix} N_{p-s} & \hat{M}_{12} \\ 0 & N_s \end{bmatrix}, \quad U_{11} Y_{11} = \begin{bmatrix} T_1 \\ 0 \end{bmatrix}, \quad U_{22} Y_{22} = \begin{bmatrix} T_2 \\ 0 \end{bmatrix},$$

where T_1 of size $(p-s) \times (p-s)$ and T_2 of size $s \times s$ are nonsingular upper triangular Toeplitz matrices. Hence s satisfies $p - r \leq s \leq \min\{p, n-r\}$. With $Z_2 := Z_1 \operatorname{diag}(Y_2 \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}^{-1}, I)$ we obtain

$$\tilde{U} := U Z_2 = \begin{array}{c} \\ p-s \\ r+s-p \\ s \\ n-r-s \end{array} \!\!\begin{array}{c} \begin{array}{ccc} p-s & s & q \end{array} \\ \begin{bmatrix} I & W_{12} & W_{13} \\ 0 & W_{22} & W_{23} \\ 0 & I & W_{33} \\ 0 & 0 & W_{43} \end{bmatrix} \end{array}, \quad \tilde{C} := Z_2^{-1} C Z_2 = \begin{array}{c} \\ p-s \\ s \\ q \end{array} \!\!\begin{array}{c} \begin{array}{ccc} p-s & s & q \end{array} \\ \begin{bmatrix} N_{p-s} & C_{12} & C_{13} \\ 0 & N_s & C_{23} \\ 0 & 0 & C_{33} \end{bmatrix} \end{array}.$$

Then, with $Z_3 := Z_2 \begin{bmatrix} I & W_{12} & W_{13} \\ 0 & I & W_{33} \\ 0 & 0 & I \end{bmatrix}^{-1}$, we have $UZ_3 = \begin{bmatrix} I & 0 & 0 \\ 0 & W_{22} & \hat{W}_{23} \\ 0 & I & 0 \\ 0 & 0 & W_{43} \end{bmatrix}$ and, since UZ_3 is nonsingular, the submatrix $Y_3 := \begin{bmatrix} \hat{W}_{23} \\ W_{43} \end{bmatrix}$ must be nonsingular. Set $Z := Z_3 \operatorname{diag}(I, Y_3^{-1})$, then

$$
V := UZ = \begin{array}{c} \\ p-s \\ r+s-p \\ s \\ n-r-s \end{array}
\begin{array}{c} \begin{array}{cccc} p-s & s & r+s-p & n-r-s \end{array} \\
\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & W_{22} & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \end{array} \tag{25}
$$

and we have that $NV = V\hat{C}$, where

$$
\hat{C} := Z^{-1}CZ = \begin{bmatrix} N_{p-s} & \hat{C}_{12} & \hat{C}_{13} \\ 0 & N_s & \hat{C}_{23} \\ 0 & 0 & \hat{C}_{33} \end{bmatrix}.
$$

This relation implies that the block W_{22} must have the form

$$
W_{22} = \begin{cases} [0,\, T] & r \le p, \\ \begin{bmatrix} T \\ 0 \end{bmatrix} & r > p, \end{cases} \tag{26}
$$

where T is an $(r+s-p)\times(r+s-p)$ or $s \times s$ upper triangular Toeplitz matrix, respectively.

With V as in (25), W_{22} satisfying (26), and a nonsingular matrix F in block upper triangular form

$$
F = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix}, \tag{27}
$$

where $F_{11} \in \mathbf{C}^{p\times p}$, $\quad F_{22} \in \mathbf{C}^{q\times q}$ and the matrix $U = VF$ always satisfies (24).

Set $\mathcal{V} = \operatorname{diag}(V, V^{-H})$ with V defined in (25). Then for the symplectic matrix $\mathcal{F} = \operatorname{diag}(F, F^{-H})$ with F nonsingular as in (27), using (21) we have the Hamiltonian triangular form

$$
\mathcal{H}(\mathcal{S}\mathcal{V}\mathcal{F}) = (\mathcal{S}\mathcal{V}\mathcal{F})\hat{\mathcal{R}},
$$

for an appropriate matrix $\hat{\mathcal{R}}$ defined as in (23). Using (22) we can now write down the bases of all different Lagrangian invariant subspaces of $\mathcal{H}$ associated with p eigenvalues λ

and q eigenvalues $-\bar{\lambda}$, as

$$\mathcal{S}\left(\begin{array}{l} \\ p-s \\ r+s-p \\ s \\ n-r-s \\ p-s \\ r+s-p \\ s \\ n-r-s \end{array}\begin{array}{c} \begin{array}{cccc} p-s & s & r+s-p & n-r-s \end{array} \\ \left[\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & W_{22} & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & -W_{22}^H & 0 \\ 0 & 0 & 0 & I \end{array}\right]\end{array}\right)X,$$

where W_{22} is as in (26) and $X =\in \mathbf{C}^{n\times n}$ is nonsingular.

It is clear that different choices of W_{22} generate different Lagrangian invariant subspaces, i.e., in this case there are infinitely many Lagrangian invariant subspaces which are parametrized via upper triangular Toeplitz matrices that are solutions to homogeneous matrix equations.

c) The characterization of all different Lagrangian invariant subspaces in this case can be obtained as in case b). Let $N = \mathrm{diag}(N_{n_1}, \ldots, N_{n_l})$. By preforming the reductions as in case b) inductively, we obtain an expression for $U = VF$, which satisfies $NU = U\left[\begin{array}{cc} \hat{M}_1 & \hat{L} \\ 0 & -\hat{M}_2^H \end{array}\right]$, where F is as in (27) and V has the form

$$\begin{array}{l} \\ s_1 \\ n_1-s_1 \\ s_2 \\ n_2-s_2 \\ \vdots \\ s_{l-1} \\ n_{l-1}-s_{l-1} \\ s_l \\ n_l-s_l \end{array}\begin{array}{c} \begin{array}{cccccccccc} s_1 & s_2 & \ldots & s_{l-1} & s_l & n_1-s_1 & n_2-s_2 & \ldots & n_{l-1}-s_{l-1} & n_l-s_l \end{array} \\ \left[\begin{array}{cccccccccc} I & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & W_{22} & \ldots & W_{2,l-1} & W_{2,l} & I & 0 & \ldots & 0 & 0 \\ 0 & I & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & W_{4,l-1} & W_{4,l} & 0 & I & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & I & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & W_{2l-2,l} & 0 & 0 & \ldots & I & 0 \\ 0 & 0 & \ldots & 0 & I & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & I \end{array}\right]\end{array}.$$

Here for $k = 1, \ldots, l$, the integers s_k satisfy $0 \le s_k \le n_k$ and $\sum_{k=1}^{l} s_k = p$ and for $i = 1, \ldots, l-1$ and $j = i+1, \ldots, l$, the blocks $W_{2i,j}$ satisfy the (singular) Sylvester equations

$$N_{n_i-s_i}W_{2i,j} - W_{2i,j}N_{s_j} = \sum_{k=i+1}^{j-1} W_{2i,k}e_{s_k}e_1^T W_{2k,j}. \tag{28}$$

The basis of an arbitrary Lagrangian invariant subspace can then be expressed as

$$\mathcal{S}\left[\begin{array}{cc} V & 0 \\ 0 & V^{-H} \end{array}\right]\mathcal{K}\left[\begin{array}{c} X \\ 0 \end{array}\right],$$

where $X \in \mathbf{C}^{n\times n}$ is nonsingular.

In general it is difficult to characterize all solutions of (28) but we can for example choose $s_l > 0$ and $W_{2i,j} = 0$ for $i = 1, \ldots, l-1$ and $j = i+1, \ldots, l-1$, then $W_{2i,l}$ has to satisfy $N_{n_i - s_i} W_{2i,l} = W_{2i,l} N_{s_l}$ for $i = 1, \ldots, l-1$. Obviously regardless whether p is fixed or not, $\mathcal{H}$ has infinitely many Lagrangian invariant subspaces associated with p copies of λ and $n-p$ copies of $-\bar{\lambda}$.

We conclude from this analysis, that only in the case a), i.e., if $\mathcal{H}$ has a single Jordan block associated with λ, we have a unique Lagrangian invariant subspace associated with p copies of λ and $n-p$ copies of $-\bar{\lambda}$. The analysis explains the observations in Example 1.

Since we can perform the same analysis for every pair λ, $-\bar{\lambda}$ with $\mathrm{Re}\,\lambda \neq 0$, we see that in order to obtain a unique Lagrangian invariant subspace, in general, every such eigenvalue can only have a single Jordan block and we have to fix the partitioning, i.e., the multiplicity p of the eigenvalue λ. If we do this for each λ separately, and the conditions for the uniqueness of the part associated with the purely imaginary eigenvalues holds, then we obtain an extension of Theorem 11.

We summarize the complete analysis in the following Theorem.

Theorem 12 *Given a Hamiltonian matrix $\mathcal{H}$ with distinct purely imaginary eigenvalues (including the eigenvalue zero) $i\alpha_1, \ldots, i\alpha_\nu$. Suppose that $\mathcal{H}$ has a Hamiltonian block triangular form.*

a) For every eigenvalue selection $\tilde{\omega}$ of n eigenvalues that is associated with a Lagrangian invariant subspace, this subspace is unique if and only if the following conditions hold:

i) All indices $\mathrm{Ind}_S^{\mathrm{c}}(i\alpha_k)$ are void and $\mathrm{Ind}_S^{\mathrm{e}}(i\alpha_k)$ contains the same structure inertia indices for all $k = 1, \ldots, \nu$.

ii) If $\tilde{\omega} \notin \Omega(\mathcal{H})$ then for each pair of eigenvalues with nonzero real part $\lambda, -\bar{\lambda} \in \tilde{\omega}$, where λ occurs with multiplicity p, there is only a single Jordan block associated with λ in the Hamiltonian Jordan form (7).

b) Let $r_1, \ldots, r_\mu$ be the multiplicities of the eigenvalues with negative real part and single Jordan blocks. Then there are

$$c := \prod_{j=1}^{\mu} (r_j + 1) \tag{29}$$

different possible eigenvalue selections which are associated with unique Lagrangian invariant subspaces.

Proof. Part a) follows from the above analysis. For part b) observe that in any case for generating a Lagrangian invariant subspace, the sum of the algebraic multiplicities of λ_j and $-\bar{\lambda}_j$ must be r_j, where r_j is the multiplicity of λ. Thus for p_j, the number of times we include λ_j, we have $r_j + 1$ different choices, $p_j \in \{0, 1, \ldots, r_j\}$. Applying this argument for all eigenvalue pairs λ_j and $-\bar{\lambda}_j$ we obtain the conclusion. $\square$

Note that if all p_j are chosen either 0 or r_j, then the associated n eigenvalues form a set contained in $\Omega(\mathcal{H})$.

We have given a complete analysis of the different possibilities of eigenvalue selections leading to unique Lagrangian invariant subspaces. For the solutions of algebraic Riccati equations we do not have such a complete characterization. We discuss some partial results in the next section.

4 Hermitian solutions of algebraic Riccati equations

We now consider Hermitian solutions of the algebraic Riccati equations

$$A^H X + XA - XDX + G = 0, \tag{30}$$

with $D = D^H$ and $G = G^H$. We have already shown that the solution is related to the Lagrangian invariant subspace of the corresponding Hamiltonian matrix $\mathcal{H} = \begin{bmatrix} A & D \\ G & -A^H \end{bmatrix}$. Unfortunately as Proposition 1 shows, the existence of a Lagrangian subspace does not imply the existence of an Hermitian solution of the algebraic Riccati equation.

Example 2 Let

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},$$

then the associated Hamiltonian matrix $\mathcal{H}$ has a unique Lagrangian invariant subspace range$\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$ but no solution to the Riccati equation exists.

Also the relaxation of the choice of eigenvalues may lead to solvability problems as the following example demonstrates.

Example 3 Let $\mathcal{H}$ be as (20) in Example 1. Then there are infinitely many solutions $X = \begin{bmatrix} 0 & \gamma \\ \bar{\gamma} & 0 \end{bmatrix}$, $\gamma \in \mathbf{C}$, corresponding to $\Lambda(A - DX) = \{1, -1\}$ but $\{1, -1\} \notin \Omega(\mathcal{H})$ and there does not exist any solution for the eigenvalue sets $\{1, 1\}, \{-1, -1\} \in \Omega(\mathcal{H})$.

The previous analysis on Lagrangian invariant subspaces now helps to analyse the properties of solutions to the algebraic Riccati equation.

Theorem 13 *Let $X = X^H$ be an Hermitian solution of (30). Let $i\alpha_1, \ldots, i\alpha_\nu$ be the pairwise distinct purely imaginary eigenvalues of $A - DX$ and let the columns of Φ_k, $k = 1, \ldots, \nu$, span the left eigenspaces corresponding to $i\alpha_k$. Suppose that $\Lambda(A - DX) \in \Omega(\mathcal{H})$. If $\Phi_k^H D \Phi_k$ is either positive definite or negative definite for all $k = 1, \ldots, \nu$, then X, as well as the associated Lagrangian invariant subspace, are unique.*

Proof. Let $\mathcal{H} = \begin{bmatrix} A & D \\ G & -A^H \end{bmatrix}$. Then, since a solution X exists, we have a symplectic matrix $\mathcal{S} = \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}$, such that

$$\mathcal{S}^{-1}\mathcal{H}\mathcal{S} = \begin{bmatrix} A - DX & D \\ 0 & -(A - DX)^H \end{bmatrix} =: \mathcal{R}.$$

By Theorem 11 and Lemma 10 the assumptions imply that $\mathcal{H}$ has a unique Lagrangian invariant subspace corresponding to $\Lambda(A - DX)$, which is just range $\begin{bmatrix} I \\ -X \end{bmatrix}$. Hence X is unique. $\square$

Theorem 13 is very useful in the context of numerical methods for the solution of algebraic Riccati equations, since it gives a criterion that can be used to check numerically the uniqueness of the solution of the Riccati equation associated with a certain spectrum.

It is clear that for the existence of an Hermitian solution of the Riccati equation we need to have conditions so that a Lagrangian invariant subspace is a graph subspace. In principle we could try all different combinations of eigenvalues, compute the Lagrangian invariant subspaces $\begin{bmatrix} U \\ V \end{bmatrix}$ and then check the invertibility of U. If there are only very few possibilities, this may be numerically feasible, but in general such a procedure will be prohibitively expensive. To derive simple necessary and sufficient conditions to guarantee that a Lagrangian invariant subspace is a graph subspace is currently an open problem. A characterization has recently been given when this is true for all Lagrangian invariant subspaces [1, 11] and there are also some special cases, see [14], where such necessary and sufficient conditions exist.

A special situation that is well understood [14, 24] is the case that D is semi-definite, which is a common condition for many realistic problems. We will slightly generalize the results given in [14, 24] using the Hamiltonian triangular form. In this way we obtain equivalent conditions that we can verify in a numerical algorithm, which is in general not possible using the conditions of [14, 24]. We need the following lemma, which is already partially shown in [14, 17] and deals with the case that (A, D) is controllable. A pair of matrices (A, B), where A is square and has the same number of rows as B, is called *controllable* if $\operatorname{rank}(\lambda I - A, B)$ is full for all complex λ.

Lemma 14 *Consider the Riccati equation (30) and let* $\mathcal{H} = \begin{bmatrix} A & D \\ G & -A^H \end{bmatrix}$. *If* (A, D) *is controllable and* D *is positive or negative semidefinite, then the following statements are equivalent.*

i) *The matrix* $\mathcal{H}$ *has a Hamiltonian block triangular form.*

ii) *For every element* $\omega \in \Omega(\mathcal{H})$ *there exists a unique Hermitian solution of (30).*

iii) *If* $i\alpha$ *is a purely imaginary eigenvalue of* $\mathcal{H}$, *then* $\operatorname{Ind}_S^d(i\alpha)$ *and* $\operatorname{Ind}_S^c(i\alpha)$ *are void, and the signs in* $\operatorname{Ind}_S^e(i\alpha)$ *are all the same.*

Proof. We assume that D is positive semidefinite, the proof for the negative semidefinite case is analogous.

i) $\Rightarrow$ **ii)** By Lemma 8 there exists a unitary symplectic matrix $\mathcal{Q} = \begin{bmatrix} Q_1 & Q_2 \\ -Q_2 & Q_1 \end{bmatrix}$, such that

$$\mathcal{Q}^H \mathcal{H} \mathcal{Q} = \mathcal{R} = \begin{bmatrix} R & K \\ 0 & -R^H \end{bmatrix}, \tag{31}$$

with $\Lambda(R) = \omega$. Employing the symplectic CS decomposition, see [20], there exist unitary matrices U, V such that

$$U^H Q_1 V = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \quad U^H Q_2 V = \begin{bmatrix} \Delta & 0 \\ 0 & I \end{bmatrix}, \tag{32}$$

where Σ and Δ are real diagonal, Σ is invertible and $\Sigma^2 + \Delta^2 = I$. For unitary symplectic matrices $\mathcal{U} = \mathrm{diag}(U, U)$ and $\mathcal{V} = \mathrm{diag}(V, V)$ we set $\hat{\mathcal{R}} := \mathcal{V}^H \mathcal{R} \mathcal{V}$, $\hat{\mathcal{H}} := \mathcal{U}^H \mathcal{H} \mathcal{U}$, $\hat{\mathcal{Q}} := \mathcal{U}^H \mathcal{Q} \mathcal{V}$. $\hat{\mathcal{Q}}$ is still unitary symplectic and from (31) we have that

$$\hat{\mathcal{Q}}^H \hat{\mathcal{H}} \hat{\mathcal{Q}} = \hat{\mathcal{R}}.$$

Let $\hat{\mathcal{H}} = \begin{bmatrix} \hat{A} & \hat{D} \\ \hat{G} & -\hat{A}^H \end{bmatrix}$ and let $\hat{A} =: \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $\hat{D} =: \begin{bmatrix} D_{11} & D_{12} \\ D_{12}^H & D_{22} \end{bmatrix}$, $\hat{G} =: \begin{bmatrix} G_{11} & G_{12} \\ G_{12}^H & G_{22} \end{bmatrix}$ be partitioned conformally with (32). If Q_1 is singular then, by comparing the corresponding blocks, we have

$$D_{22} = 0, \quad A_{21}\Sigma = D_{12}^H \Delta.$$

Since $\hat{D} \geq 0$, $D_{22} = 0$ implies $D_{12} = 0$ and since $\det \Sigma \neq 0$ we have $A_{21} = 0$ and then

$$\hat{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} D_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$

This implies that $(\hat{A}, \hat{D})$ is not controllable, which is a contradiction. Hence we obtain that Q_1 is invertible and $X = -Q_2 Q_1^{-1}$ solves (30) with $\Lambda(A - DX) = \Lambda(R) = \omega$.

It remains to prove the uniqueness of the solution. Since a solution X exists, with $\mathcal{S} = \begin{bmatrix} I & 0 \\ -X & I_n \end{bmatrix}$ we obtain (4), i.e.,

$$\mathcal{H}\mathcal{S} = \mathcal{S} \begin{bmatrix} A - DX & D \\ 0 & -(A - DX)^H \end{bmatrix}.$$

Suppose that $i\alpha$ is a purely imaginary eigenvalue of $A - DX$ and suppose that the columns of Φ span the corresponding left eigenvector subspace. If there is a nonzero vector $\phi \in \mathrm{range}\,\Phi$ such that $\phi^H D \phi = 0$ then, since $D \geq 0$, we have $\phi^H D = 0$. On the other hand ϕ is a left eigenvector of $A - DX$ so

$$i\alpha\phi^H = \phi^H (A - DX) = \phi^H A.$$

This implies that (A, D) is not controllable which is a contradiction. Therefore we have $\Phi^H D\Phi > 0$ and by Theorem 13, the solution X, as well as the related Lagrange invariant subspace, is unique.

ii) $\Rightarrow$ **iii)** From ii) we have that for each $\omega \in \Omega(\mathcal{H})$ there exists a unique Lagrangian invariant subspace. By Lemma 10, $\mathrm{Ind}_s^d(i\alpha)$ and $\mathrm{Ind}_S^c(i\alpha)$ must be void. Since $D \geq 0$, again from Lemma 10 the indices $\mathrm{Ind}_S^e(i\alpha)$ are all equal to 1.

iii) $\Rightarrow$ **i)** follows directly from Theorem 5. $\Box$

It should be noted that if $D \geq 0$ and (A, D) is controllable, then Lemma 14 iii) implies that $\mathcal{H}$ has a Hamiltonian block triangular form if and only if the Jordan blocks to all imaginary eigenvalues have even size. This condition was used in [14]. The equivalent condition that we have used here, that $\mathcal{H}$ has a Hamiltonian triangular form has the advantage, that it is easier to verify in a numerical algorithm by using the invariant subspace conditions of Theorem 5.

For the matrix triple (A, D, G) corresponding to the Riccati equation (30), using unitary similarity transformations, we have the condensed form,

$$\begin{aligned}
\hat{A} &= U^H AU = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \\
\hat{D} &= U^H DU = \begin{bmatrix} D_{11} & 0 \\ 0 & 0 \end{bmatrix}, \\
\hat{G} &= U^H GU = \begin{bmatrix} G_{11} & G_{12} \\ G_{12}^H & G_{22} \end{bmatrix},
\end{aligned} \tag{33}$$

where (A_{11}, D_{11}) is controllable. This condensed form can be computed in a numerically stable way using the staircase algorithm in [22]. It is then obvious that X is a solution of (30) if and only if $U^H XU$ is a solution of the Riccati equation with coefficients $(\hat{A}, \hat{D}, \hat{G})$ and for the closed loop spectra we have $\Lambda(A - DX) = \Lambda(\hat{A} - \hat{D}\hat{X})$. Hence in what follows without loss of generality we may assume that A, D, G are in the condensed form (33). The next result shows that there exists a Hamiltonian block triangular form for

$$H := \begin{bmatrix} A_{11} & D_{11} \\ G_{11} & -A_{11}^H \end{bmatrix} \tag{34}$$

if and only if there exists one for

$$\mathcal{H} = \begin{bmatrix} A & D \\ G & -A^H \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & D_{11} & 0 \\ 0 & A_{22} & 0 & 0 \\ G_{11} & G_{12} & -A_{11}^H & 0 \\ G_{12}^H & G_{22} & -A_{12}^H & -A_{22}^H \end{bmatrix}. \tag{35}$$

Lemma 15 *Let $\mathcal{H}$ and H be in (35) and (34). Then $\mathcal{H}$ has a Hamiltonian block triangular form if and only if H has a Hamiltonian block triangular form.*

Proof. Introduce the unitary (but non-symplectic) matrix

$$\mathcal{Z} = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \\ -I & 0 & 0 & 0 \end{bmatrix}.$$

Then with $B := \begin{bmatrix} A_{12} \\ G_{12} \end{bmatrix}$, we have that

$$\hat{\mathcal{H}} := \mathcal{Z}^H \mathcal{H} \mathcal{Z} = \begin{bmatrix} -A_{22}^H & B^H J & -G_{22} \\ 0 & H & B \\ 0 & 0 & A_{22} \end{bmatrix},$$

is block upper triangular. If there is a purely imaginary eigenvalue $i\alpha \in \Lambda(H)$, then $i\alpha \in \Lambda(\mathcal{H})$. Let Y_{11}, Y_{22}, Y_{33} be bases of the right invariant subspaces of $-A_{22}^H, H, A_{22}$ corresponding to $i\alpha$, respectively. (Note that if $i\alpha \notin \Lambda(A_{22})$ then $i\alpha \notin \Lambda(-A_{22}^H)$ and Y_{11} and Y_{33} are void.) Then we can determine a matrix

$$Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ 0 & Y_{22} & Y_{23} \\ 0 & 0 & Y_{33} \end{bmatrix},$$

such that its columns form a basis of the invariant subspace of $\hat{\mathcal{H}}$ corresponding to $i\alpha$ and hence the columns of $U := \mathcal{Z}Y$ form a basis of the corresponding invariant subspace of $\mathcal{H}$. A simple calculation yields that

$$U^H J U = \begin{bmatrix} 0 & 0 & Y_{11}^H Y_{33} \\ 0 & Y_{22}^H J Y_{22} & E_{23} \\ -Y_{33}^H Y_{11} & -E_{23}^H & E_{33} \end{bmatrix},$$

with $E_{33} = -E_{33}^H$. Note that by construction the columns of Y_{33} form a basis of the left invariant subspace of $-A_{22}^H$ corresponding to $i\alpha$. Hence $\det(Y_{11}^H Y_{33}) \neq 0$ and we can form the nonsingular matrix

$$T := \begin{bmatrix} I & \frac{1}{2}(Y_{11}Y_{33})^{-H} E_{33} (Y_{11}^H Y_{33})^{-1} & -(Y_{11}^H Y_{33})^{-H} E_{23}^H \\ 0 & 0 & I \\ 0 & (Y_{11}^H Y_{33})^{-1} & 0 \end{bmatrix}.$$

Then we get

$$(UT)^H J U T = \begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & Y_{22}^H J Y_{22} \end{bmatrix}$$

and hence $(UT)^H J (UT)$ is congruent to J if and only if $Y_{22}^H J Y_{22}$ is congruent to J. Applying Theorem 5 finishes the proof. □

Note that in this lemma we have not assumed controllability of (A, D) nor that D is semidefinite.

Using the condensed form (33), it is clear that $X := \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^H & X_{22} \end{bmatrix}$ (partitioned analogous to (33)) is an Hermitian solution of (30) with $\Lambda(A - DX) = \omega \in \Omega(\mathcal{H})$ if and only if X_{11} is an Hermitian solution of

$$A_{11}^H X_{11} + X_{11} A_{11} - X_{11} D_{11} X_{11} + G_{11} = 0, \tag{36}$$

X_{12} is a solution of the Sylvester equation

$$(A_{11} - D_{11} X_{11})^H X_{12} + X_{12} A_{22} + X_{11} A_{12} + G_{12} = 0, \tag{37}$$

X_{22} is an Hermitian solution of the Lyapunov equation

$$A_{22}^H X_{22} + X_{22} A_{22} + G_{22} + A_{12}^H X_{12} + X_{12}^H A_{12} - X_{12}^H D_{11} X_{12} = 0, \tag{38}$$

and

$$\Lambda(A - DX) = \Lambda(A_{11} - D_{11} X_{11}) \cup \Lambda(A_{22}) = \omega.$$

We have already characterized the solvability of (36), solvability conditions for (37) and (38) are well known, see, e.g., [13]. We thus obtain the following theorem.

Theorem 16 *Consider the ARE (30) with A, D, G in condensed form (33). Suppose that $\mathcal{H} = \begin{bmatrix} A & D \\ G & -A^H \end{bmatrix}$ has a Hamiltonian triangular form, (A_{11}, D_{11}) is controllable and D_{11} is semidefinite. Then the algebraic Riccati equation (30) has an Hermitian solution X with $\Lambda(A - DX) = \omega$ if and only if $\Lambda(A_{22}) \subset \omega$ and there exist solutions for the Sylvester equation (37) and the Lyapunov equation (38).*

If an Hermitian solution exists and A_{22} has no purely imaginary eigenvalue then this solution is unique; otherwise there are infinitely many solutions.

Proof. The necessary and sufficient condition is obvious. We only need to prove the uniqueness.

Under the given conditions by Lemma 14 and Lemma 15 there exists a unique Hermitian solution of (36) and $\Lambda(A_{11} - D_{11} X_{11}) \subset \omega$. If A_{22} has no purely imaginary eigenvalues then, since $\Lambda(A_{11} - D_{11} X_{11}) \cup \Lambda(A_{22}) = \omega \in \Omega(\mathcal{H})$, we have $\Lambda((A_{11} - D_{11} X_{11})^H) \cap \Lambda(-A_{22}) = \emptyset$. Similarly $\Lambda(A_{22}^H) \cap \Lambda(-A_{22}) = \emptyset$. Since X_{11} is already uniquely determined, the solvability theory for Sylvester and Lyapunov equations, (see [13]) yields that the solutions X_{12}, X_{22} of (37) and (38) are also uniquely determined.

If A_{22} has some purely imaginary eigenvalues and if (38) has a solution, then there exist infinitely many solutions. Moreover if $\Lambda(A_{11} - D_{11} X_{11}) \cap \Lambda(A_{22}) \neq \emptyset$ and equation (37) has a solution, then it also has infinitely many solutions. □

This result is a slight generalization of the results in [24] and [14, Theorem 7.9.1; Lemma 7.9.6]. Unlike the case in Lemma 14, where the solution always exists and is unique for each $\omega \in \Omega(\mathcal{H})$, here the existence of the solution depends on the choice of ω. This means that solutions only exist for a subset of $\Omega(\mathcal{H})$. Moreover, when solutions exist the uniqueness depends on the spectrum of A_{22}, i.e., whether A_{22} has purely imaginary eigenvalues or not.

If in Theorem 16, both D and G are positive semidefinite or negative semidefinite, then the result can be further improved.

Theorem 17 *Let* $\mathcal{H} = \begin{bmatrix} A & D \\ G & -A^H \end{bmatrix}$ *with* D, G *both positive semidefinite or both negative semidefinite and* (A, D, G) *in the condensed form of (33). If* (A_{11}, D_{11}) *is controllable, then* $\mathcal{H}$ *has a Hamiltonian block triangular form.*

Proof. By Lemma 15 we only need to prove that $H = \begin{bmatrix} A_{11} & D_{11} \\ G_{11} & -A_{11}^H \end{bmatrix}$ has a Hamiltonian block triangular form and (by possibly multiplying the matrix by -1) we may assume without loss of generality that $D, G \geq 0$, which implies that also $D_{11}, G_{11} \geq 0$. Since we can apply the transformation to condensed form (33) also to the triple $(A_{11}^H, G_{11}, D_{11})$, we may assume without loss of generality that

$$A_{11} = \begin{bmatrix} \hat{A}_{11} & 0 \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \; D_{11} = \begin{bmatrix} \hat{D}_{11} & \hat{D}_{12} \\ \hat{D}_{12}^H & \hat{D}_{22} \end{bmatrix}, \; G_{11} = \begin{bmatrix} \hat{G}_{11} & 0 \\ 0 & 0 \end{bmatrix},$$

where $(\hat{A}_{11}^H, \hat{G}_{11})$ is controllable. Moreover, since D_{11} is positive semidefinite, the controllability of (A_{11}, D_{11}) implies that $(\hat{A}_{11}, \hat{D}_{11})$ is also controllable. It is well-known, see, e.g., [20] that under these assumptions the Hamiltonian matrix

$$\hat{H} = \begin{bmatrix} \hat{A}_{11} & \hat{D}_{11} \\ \hat{G}_{11} & -\hat{A}_{11}^H \end{bmatrix}$$

has no purely imaginary eigenvalue. By Theorem 5 there exists a symplectic matrix

$$\hat{S} = \begin{bmatrix} \hat{S}_{11} & \hat{S}_{12} \\ \hat{S}_{21} & \hat{S}_{22} \end{bmatrix}$$

such that $\hat{S}^{-1}\hat{H}\hat{S}$ is Hamiltonian block triangular. Therefore with the extended symplectic matrix

$$S = \begin{bmatrix} \hat{S}_{11} & 0 & \hat{S}_{12} & 0 \\ 0 & I & 0 & 0 \\ \hat{S}_{21} & 0 & \hat{S}_{22} & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

we obtain that $S^{-1}HS$ is Hamiltonian block triangular. $\Box$

Since the condensed form (33) can be constructed for arbitrary matrix triples (A, D, G) we immediately have the following corollary.

Corollary 18 *Every Hamiltonian matrix* $\mathcal{H} = \begin{bmatrix} A & D \\ G & -A^H \end{bmatrix}$ *with* $D, G \geq 0$ *or* $D, G \leq 0$ *has a Hamiltonian triangular form.*

We also directly obtain a solvability result for the related Riccati equation (30).

Corollary 19 *Consider the Riccati equation (30) with* (A, D, G) *in condensed form as (33). Suppose that* $D, G \geq 0$ *or* $D, G \leq 0$. *Then (30) has an Hermitian solution* X *with* $\Lambda(A - DX) = \omega \in \Omega(\mathcal{H})$ *if and only if* $\Lambda(A_{22}) \subset \omega$ *and there exist solutions for the Sylvester equation (37) and the Lyapunov equation (38), where* X_{11} *is the unique solution of (36) satisfying* $\Lambda(A_{11} - D_{11}X_{11}) \subset \omega$.

If an Hermitian solution exists and A_{22} *has no purely imaginary eigenvalues then it is unique, otherwise infinitely many solutions exist.*

Proof. By Theorem 17 the Hamiltonian matrix $\mathcal{H}$ has a Hamiltonian block triangular form. The result then follows directly from Theorem 16. □

In all the presented results concerning Lagrangian invariant subspaces we can exchange the roles of D, G, while for the solvability theory of the Riccati equation in general this is not possible.

If $G \geq 0$, then we can reduce the problem by splitting $D = D_1 - D_2$ with D_1, D_2 positive semidifinite. For example, in H_∞ control we need to solve algebraic Riccati equations of the type

$$A^H X + XA - X(D_1 - \gamma^{-2} D_2)X + G = 0, \tag{39}$$

where D_1, D_2, G are all Hermitian and positive semidefinite. Under mild assumptions [8], for γ large enough the related Hamiltonian matrix $\mathcal{H} = \begin{bmatrix} A & D_1 - \gamma^{-2} D_2 \\ G & -A^H \end{bmatrix}$ always has a Hamiltonian block triangular form and the solution of the dual Riccati equation

$$AX + XA^H - XGX + D_1 - \gamma^{-2} D_2 = 0$$

always exists. However the solution of (39) may not exist for the same γ.

To analyse this problem, it is well-known, see e.g., [9] that one can split the problem and first transform $\mathcal{H}$ to

$$\mathcal{H}_0 = \begin{bmatrix} A - X_0 G & -\gamma^{-2} D_2 \\ G & -(A - X_0 G)^H \end{bmatrix}$$

with a symplectic matrix $\begin{bmatrix} I & X_0 \\ 0 & I \end{bmatrix}$, where X_0 is an Hermitian solution of the Riccati equation

$$AX + XA^H - XGX + D_1 = 0.$$

By the presented theory we have the solvability theory for this subproblem and thus we have reduced the analysis to the Riccati equation

$$(A - X_0 G)^H Y + Y(A - X_0 G) - \gamma^{-2} Y D_2 Y - G = 0,$$

which is in the form (30). If a solution Y for this equation exists and $I + X_0 Y$ is nonsingular then $-Y(I + X_0 Y)^{-1}$ solves (39). In principle such a process can be repeated but it is an open problem, whether a complete solvability theory can be developed this way.

5 Conclusion

Based on the recently developed canonical forms for Hamiltonian matrices under symplectic similarity transformations we have given necessary and sufficient conditions for the existence and uniqueness of Lagrangian invariant subspaces. Similarly, but only in special cases, we have discussed the existence and uniqueness of solutions to the related algebraic Riccati equation. The general case is still an open problem.

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