NODA ITERATIONS FOR GENERALIZED EIGENPROBLEMS FOLLOWING PERRON-FROBENIUS THEORY*

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Abstract. In this paper, we investigate the generalized eigenvalue problem $A\mathbf{x} = \lambda B\mathbf{x}$ arising from economic models. Under certain conditions, there is a simple generalized eigenvalue $\rho(A, B)$ in the interval $(0, 1)$ with a positive eigenvector. Based on the Noda iteration, a modified Noda iteration (MNI) and a generalized Noda iteration (GNI) are proposed for finding the generalized eigenvalue $\rho(A, B)$ and the associated unit positive eigenvector. It is proved that the GNI method always converges and has a quadratic asymptotic convergence rate. So GNI has a similar convergence behavior as MNI. The efficiency of these algorithms is illustrated by numerical examples.

Key words. Generalized eigenproblem, generalized Noda iteration, nonnegative irreducible matrix, M-matrix, quadratic convergence, Perron-Frobenius theory.

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1. Introduction. For any real matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, we say that A is nonnegative, and write $A \geq 0$, if $a_{ij} \geq 0$ for all i, j. The matrix A is called positive, $A > 0$, if $a_{ij} > 0$ for all i, j. If $A = [a_{ij}], B = [b_{ij}]$ are $n \times n$ real matrices, then $A \geq B(A > B)$ means that $a_{ij} \geq b_{ij}(a_{ij} > b_{ij})$ for all i, j. A nonnegative (positive) vector is defined in the same way. The identity matrix of an appropriate size is denoted by I. For a matrix A, its transpose is denoted by A^T. $\|\cdot\|$ denotes the Euclidean vector norm and the matrix spectral norm. For real vectors $\mathbf{v} = \left(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(n)}\right)^T$ and $\mathbf{w} = \left(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \ldots, \mathbf{w}^{(n)}\right)^T$ with $\mathbf{v}^{(i)} \neq 0$ for all i, we use $\frac{\mathbf{w}}{\mathbf{v}}$ to denote the column vector whose ith component is $\frac{\mathbf{w}^{(i)}}{\mathbf{v}^{(i)}}$. We also define $\max \mathbf{w} = \max\limits_i \mathbf{w}^{(i)}$ and $\min \mathbf{w} = \min\limits_i \mathbf{w}^{(i)}$. An $n \times n$ matrix A is said to be *reducible* if A is either the 1×1 zero matrix or if there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} E & 0 \\ F & G \end{bmatrix},$$

where E, G are square matrices. If A is not reducible, then it is called *irreducible*. A matrix pair (A, B) is *regular* if both A and B are $n \times n$ and there exists a pair of complex numbers $(\alpha, \beta) \neq (0, 0)$ such that $\det(\alpha A + \beta B) \neq 0$, [6, 16].

The generalized eigenproblem of a general matrix pair (A, B) with $A, B \in \mathbb{C}^{m\times n}$ is to solve the equation:

$$A\mathbf{x} = \lambda B\mathbf{x}, \quad \lambda \in \mathbb{C},\ 0 \neq \mathbf{x} \in \mathbb{C}^n. \tag{1.1}$$

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If λ and $\mathbf{x}$ satisfies this equation, we call $(\lambda, \mathbf{x})$ an *eigenpair* of (A, B). When $B = I$, it is an eigenpair of the matrix A.

The eigenproblem (1.1) has been extensively studied and various powerful numerical methods have been developed, e.g., [6, 15, 16]. In this paper, we consider the eigenvalue problem of a special type of matrix pairs that exhibit the same kind of properties of nonnegative matrices provided by the Perron-Frobenius theory. Such a generalized eigenproblem has important applications in economy in the context of Sraffa's model for the joint production of commodities by means of commodities, see [14, page 53] and [13], where the entry a_{ij} of A indicates the quantity of the ith commodity that enters as means of production into the jth industry, and the entry b_{ij} of B indicates the quantity of ith commodity produced by the jth industry. A closely related generalized eigenproblem also arises from finite element approximation of the eigenvalue problem of second-order elliptic operators [7] (also see Example 5.3).

Motivated by a nonlinear extension of the Perron-Frobenius theory in [11], Fujimoto considered in [5] the problem (1.1) with $A, B \in \mathbb{R}^{n\times n}$ and satisfying the following conditions:

(C1) $A \geq 0$.
(C2) A is irreducible.
(C3) there exists a vector $\mathbf{v} > 0$ such that $B\mathbf{v} > A\mathbf{v}$.
(C4) for all $i \neq j$, $b_{ij} \leq a_{ij}$.

Economic interpretation of these conditions is given in [5]. It has shown in [5] that if the conditions (C1)-(C4) are satisfied, then (1.1) has a solution $(\lambda, \mathbf{x})$ with $\lambda \in (0, 1)$ and $\mathbf{x} > 0$. In [2], a stronger result was derived and it is contained in the following theorem.

THEOREM 1.1. *Let A and B be $n \times n$ matrices satisfying the conditions (C1)-(C4). Then there exist $\lambda \in (0, 1)$ and a vector $\mathbf{x}_* > 0$ such that $A\mathbf{x}_* = \lambda B\mathbf{x}_*$.*

Furthermore, if $A\mathbf{v} = \lambda' B\mathbf{v}$ for $\lambda' \geq 0$ and $\mathbf{v} \geq 0, \mathbf{v} \neq 0$, then $\lambda' = \lambda$ and $\mathbf{v} = \alpha\mathbf{x}_*$ for some $\alpha > 0$.

Because of Theorem 1.1, we are able to give the following definition.

DEFINITION 1.1. *Under the conditions (C1)-(C4), we call the unique value $\lambda \in (0, 1)$ in Theorem 1.1 the* Perron root *of the pair (A, B) and denote it by $\rho(A, B)$, and its corresponding positive eigenvector $\mathbf{x}_*$ is called the* Perron vector *of (A, B). We call the pair $(\rho(A, B), \mathbf{x}_*)$ the Perron pair of (A, B).*

If A is a nonnegative irreducible matrix and $B = I$, then Theorem 1.1 reduces to the Perron-Frobenius Theorem of nonnegative irreducible matrices ([3, Theorem 1.4]). In this case $\rho(A, I) = \rho(A)$, the Perron root of A. Therefore, $\rho(A, B)$ can be thought of as a generalization of the Perron root of a nonnegative irreducible matrix. The difference is, unlike the Perron root of nonnegative matrix, $\rho(A, B)$ may not necessarily be the largest generalized eigenvalue of $A\mathbf{x} = \lambda B\mathbf{x}$ in magnitude ([2]).

In 1971, Noda [12] introduced a positivity preserving method, which is now called the *Noda iteration (NI)*, for computing the Perron pair (Perron root and Perron vector) of a nonnegative matrix. NI always converges and has a quadratic convergence rate ([4, 8]). Recently, NI has be modified for solving the eigenproblem of M-matrices and nonnegative tensors, e.g., [8, 9, 10, 18]. In this paper, we propose two NI type iterative methods for computing the Perron pair of a matrix pair (A, B) satisfying the conditions (C1)-(C4). We first present a modified Noda iteration (MNI) that is directly modified from the standard NI. Next, we propose a generalized Noda iteration (GNI). GNI is similar to MNI. Both can be viewed as shift-invert power (SIP) methods ([1]). Unlike a general SIP, both GNI and MNI have three advantages: (i) The shifts generated from the iteration form a monotonically decreasing sequence and will converge to the Perron root from the right; (ii) For an arbitrary positive initial vector, the vectors generated from the iteration preserve the positivity and will converge to a Perron vector; (iii) The asymptotic convergence rate is quadratic.

The paper is organized as follows. In Section 2, we further give some properties of the Perron pair for the matrix pair (A, B) satisfying the conditions (C1)-(C4). In Section 3, we introduce the Noda iteration and propose a modified Noda iteration (MNI) and a generalized Noda iteration (GNI) for the matrix pair satisfying the conditions (C1)-(C4). In Section 4, we establish a convergence theory for GNI. There we show that GNI always converges at a quadratic rate. In Section 5, we present two numerical examples to demonstrate the effectiveness and convergence behavior of both iterations. Some concluding remarks are given in Section 6.

2. Generalized eigenproblem satisfying the conditions (C1)-(C4). In this section, we further present some existing and new properties of the Perron pair for a matrix pair that satisfies the conditions (C1)-(C4). The properties will be served as a basis for constructing the iterative methods and studying their convergence behavior.

First, we review some fundamental properties related to nonnegative matrices. A real square matrix A is a *Z-matrix* if all its off-diagonal elements are nonpositive. Any Z-matrix can be written as $sI - B$ with $B \geq 0$, and it is a nonsingular *M-matrix* if $s > \rho(B)$. Here $\rho(B)$ is the Perron root of B.

The following result is well known [17, 19].

PROPOSITION 2.1. *For a Z-matrix A, the following are equivalent:*

(1) *A is a nonsingular M-matrix.*

(2) $A^{-1} \geq 0$.

(3) $A\mathbf{v} > 0$ *for some vector* $\mathbf{v} \geq 0$.

In [2], by using Proposition 2.1 (3) it shows that if a matrix pair (A, B) satisfies the conditions (C3) and (C4) then the matrix $B - A$ is an invertible M-matrix. This also implies that under the conditions (C3) and (C4) the matrix pair (A, B) is regular.

When $B - A$ is invertible, the generalized eigenproblem (1.1) is equivalent to the standard eigenproblem of the matrix $(B - A)^{-1}A$. This is the fundamental transformation used in [2] for generalizing the Perron-Frobenius theory. We summarize the relations between these two eigenproblems in the following lemmas and theorems. Most of the results can be found in [2].

LEMMA 2.1. *Let (A, B) be an $n \times n$ matrix pair and let $B - A$ be invertible. Then $(\lambda, \mathbf{x})$ is an eigenpair of the matrix $(B - A)^{-1}A$ if and only if $(\frac{\lambda}{1+\lambda}, \mathbf{x})$ is an eigenpair of (A, B).*

Proof. The proof is straightforward. $(\lambda, \mathbf{x})$ is an eigenpair of $(B - A)^{-1}A$ if and only if $(B - A)^{-1}A\mathbf{x} = \lambda\mathbf{x}$. The latter is equivalent to

$$A\mathbf{x} = \lambda(B - A)\mathbf{x},$$

which is further equivalent to

$$(1+\lambda)A\mathbf{x} = \lambda B\mathbf{x} \quad \text{or} \quad A\mathbf{x} = \frac{\lambda}{1+\lambda}B\mathbf{x}.$$

□

COROLLARY 2.1. *For a matrix pair (A, B) under the assumptions of Lemma 2.1, define $C = (B-A)^{-1}A$. Let λ be an eigenvalue of C. Then $\rho = \lambda/(1+\lambda)$ is an eigenvalue of (A, B). The relation between λ and ρ can be characterized as follows.*

(1) $\lambda \geq 0$ *if and only if* $\rho \in [0, 1)$*;*

(2) $-1 \neq \lambda < 0$ *if and only if* $\rho \in (-\infty, 0) \cup (1, +\infty)$ *;*

(3) $\lambda = -1$ *if and only if* $\rho = \infty$*;*

(4) λ *is complex and* $\mathrm{Im}(\lambda) \neq 0$ *if and only if* ρ *is complex and* $\mathrm{Im}(\rho) \neq 0$*, where* $\mathrm{Im}(z)$ *denotes the imaginary part of a complex number* z*.*

REMARK 2.1. When $\lambda = -1$, we have $B\mathbf{x} = 0$ for some nonzero vector $\mathbf{x}$. In that case, the matrix pair (A, B) is said to have an infinite eigenvalue, see e.g., [Chap. VI, 16].

The main results in this section are included in the following theorems.

THEOREM 2.1 ([2]). *Let (A, B) be an $n \times n$ matrix pair satisfying the conditions (C1)-(C4), and let $C = (B-A)^{-1}A$. Then C is an irreducible nonnegative matrix, and we have the follow properties:*

(1) $\rho(A, B) = \frac{\rho(C)}{1+\rho(C)} \in (0, 1)$ *is a simple eigenvalue of (A, B) with a positive eigenvector $\mathbf{x}_*$;*

(2) *Any nonnegative eigenvector of (A, B) must be $\alpha\mathbf{x}_*$ for some $\alpha > 0$;*

(3) *There is no eigenvalue of (A, B) lying in the interval $(\rho(A, B), 1]$.*

Proof. The results are simply from [2, Theorem 2.2] and Corollary 2.1. □

Note that Theorem 2.1 includes Theorem 1.1.

THEOREM 2.2. *Let (A, B) be an $n \times n$ matrix pair satisfying the conditions (C1)-(C4). Then for any $\rho \in (\rho(A, B), 1]$, $\rho B - A$ is a nonsingular M-matrix.*

Proof. Since $\lambda \leq 1$, from the condition (C3), $\lambda B - A$ is a Z-matrix. Based on Theorem 2.1, there exists a Perron vector $\mathbf{x}_* > 0$ such that

$$(\rho(A, B)B - A)\mathbf{x}_* = 0.$$

Since A is nonnegative and irreducible, $B\mathbf{x}_* = \frac{1}{\rho(A,B)}A\mathbf{x}_* > 0$. Because $\rho > \rho(A, B)$, we get

$$(\rho B - A)\mathbf{x}_* = (\rho - \rho(A, B))B\mathbf{x}_* + (\rho(A, B)B - A)\mathbf{x}_* = (\rho - \rho(A, B))B\mathbf{x}_* > 0.$$

So by Proposition 2.1 (3), $\rho B - A$ is a nonsingular M-matrix. □

The following result is a generalization of [17, Theorem 2.2] and [2, Theorem 3.2].

THEOREM 2.3. *Let (A, B) be an $n \times n$ matrix pair satisfying the conditions (C1)-(C4). If $\mathbf{v} \geq 0$ is not an generalized eigenvector of (A, B) and satisfies $B\mathbf{v} > 0$,* *then*

$$\min \frac{A\mathbf{v}}{B\mathbf{v}} < \rho(A,B) < \max \frac{A\mathbf{v}}{B\mathbf{v}}. \tag{2.1}$$

Proof. The proof follows directly from the proofs for Theorem 3.2 and Corollary 3.3 in [2]. The inequalities must be strict, since one of them becomes equality only when $\min \frac{A\mathbf{v}}{B\mathbf{v}} = \max \frac{A\mathbf{v}}{B\mathbf{v}}$. The latter holds true if and only if $\mathbf{v}$ is an eigenvector of (A,B). □

The following result is from [2, Corollary 3.3].

THEOREM 2.4. *Under the conditions (C1)-(C4), we have*

$$\max_{\mathbf{v}\geq 0, B\mathbf{v}>0} \min \frac{A\mathbf{v}}{B\mathbf{v}} = \rho(A,B) = \min_{\mathbf{v}\geq 0, B\mathbf{v}>0} \max \frac{A\mathbf{v}}{B\mathbf{v}}.$$

3. Noda iterations for generalized eigenproblems.

The original Noda iteration (NI) is for computing the Perron pair of an irreducible nonnegative matrix C ([12]).

Algorithm 1. Noda iteration (NI)

1. Given $\lambda_0 > \rho(C), \mathbf{x}_0 > 0$ with $\|\mathbf{x}_0\| = 1$, and tol > 0.
2. **for** $k = 0, 1, 2, \dots$
3. Solve $(\lambda_k I - C)\mathbf{y}_{k+1} = \mathbf{x}_k$.
4. Normalize the vector $\mathbf{x}_{k+1} = \mathbf{y}_{k+1}/\|\mathbf{y}_{k+1}\|$.
5. Compute $\lambda_{k+1} = \lambda_k - \min \frac{\mathbf{x}_k}{\mathbf{y}_{k+1}}$.

   until $\|\lambda_{k+1}\mathbf{x}_{k+1} - C\mathbf{x}_{k+1}\| <$ tol.
6. Output: $\rho(C) \leftarrow \lambda_{k+1}$ and $\mathbf{x}_* \leftarrow \mathbf{x}_{k+1}$.

The algorithm NI is similar to the Rayleigh quotient iteration, but with a different way of choosing shifts. In fact, $\lambda_{k+1} = \max(C\mathbf{x}_{k+1}/\mathbf{x}_{k+1})$, and from the linear system in the iteration, one has

$$\lambda_{k+1} = \max \frac{C\mathbf{x}_{k+1}}{\mathbf{x}_{k+1}} = \max \frac{C\mathbf{y}_{k+1}}{\mathbf{y}_{k+1}} = \max \frac{\lambda_k \mathbf{y}_{k+1} - \mathbf{x}_k}{\mathbf{y}_{k+1}} = \lambda_k - \min \frac{\mathbf{x}_k}{\mathbf{y}_{k+1}}.$$

Clearly, if $\mathbf{x}_k > 0$ and $\lambda_k > \rho(C)$, because $\lambda_k I - C$ in an invertible M-matrix, we have $\mathbf{y}_{k+1} > 0$ and $\mathbf{x}_{k+1} > 0$. Then $\lambda_{k+1} < \lambda_k$, and from [17, Theorem 2.2], $\lambda_{k+1} > \rho(C)$. Therefore, once λ_0 and $\mathbf{x}_0$ satisfy the conditions in Algorithm 1, NI generates a decreasing shift sequence $\{\lambda_k\}$ bounded below from $\rho(C)$ and a positive vector sequence $\{\mathbf{x}_k\}$. It is known that $(\{\lambda_k\}, \{\mathbf{x}_k\})$ always converge to the Perron pair of the matrix C quadratically ([4]).

For a matrix pair (A,B) satisfying the conditions (C1)-(C4), based on Theorem 2.1 its Perron pair can be computed by applying Algorithm 1 directly to the irreducible nonnegative matrix $C := (B-A)^{-1}A$. However, computing C explicitly may cause numerical instability when $B-A$ is ill-conditioned. In order to avoid this, we pre-multiply $B-A$ to the linear system in Step 3 of Algorithm 1 and change it to

$$[\lambda_k B - (1+\lambda_k)A]\,\mathbf{y}_{k+1} = (B-A)\mathbf{x}_k.$$

We also change the shift λ_k to $\rho_k := \lambda_k/(1+\lambda_k)$. By dividing $1+\lambda_k$ on both sides of the above equation and using $\frac{1}{1+\lambda_k} = 1-\rho_k$, we obtain

$$(\rho_k B - A)\mathbf{y}_{k+1} = (1-\rho_k)(B-A)\mathbf{x}_k. \tag{3.1}$$

With $\tau_k := \min(\mathbf{x}_k/\mathbf{y}_{k+1})$, from Step 5 of Algorithm 1, $\lambda_{k+1} = \lambda_k - \tau_k$. Then

$$\begin{aligned}\rho_{k+1} &= \frac{\lambda_k - \tau_k}{1+(\lambda_k-\tau_k)} = \frac{\lambda_k/(1+\lambda_k) - \tau_k/(1+\lambda_k)}{1-\tau_k/(1+\lambda_k)} \\ &= \frac{\rho_k - (1-\rho_k)\tau_k}{1-(1-\rho_k)\tau_k} = \rho_k - \frac{(1-\rho_k)^2\tau_k}{1-(1-\rho_k)\tau_k}. \end{aligned} \tag{3.2}$$

This formula can be used for computing the next shift during the iteration process. Since $\{\lambda_k\}$ generated in Algorithm 1 is decreasing and since the function $\rho = \lambda/(1+\lambda)$ is an increasing function in the interval $[0,\infty]$, $\{\rho_k\}$ is decreasing as well. If the initial shift ρ_0 is selected on the interval $(\rho(A,B),1)$, which is equivalent to, $\lambda_0 > \rho(C)$, the shift sequence $\{\rho_k\}$ is decreasing and converges to $\rho(A,B)$ from the right. We may simplify the iteration further based on the the fact that in each iteration we only need the direction of $\mathbf{y}_{k+1}$. We may remove the scalar $1-\rho_k$ on the right hand side of (3.1) by replacing $\mathbf{y}_{k+1}$ with $\tilde{\mathbf{y}}_{k+1} := \mathbf{y}_{k+1}/(1-\rho_k)$. Then (3.1) becomes

$$(\rho_k B - A)\tilde{\mathbf{y}}_{k+1} = (B-A)\mathbf{x}_k.$$

By introducing

$$\tilde{\tau}_k := (1-\rho_k)\tau_k = \min\frac{\mathbf{x}_k}{\tilde{\mathbf{y}}_{k+1}},$$

the formula (3.2) becomes

$$\rho_{k+1} = \rho_k - \frac{(1-\rho_k)\tilde{\tau}_k}{1-\tilde{\tau}_k}.$$

We then have the following modified version (MNI) of NI that applies directly to the matrix pair (A,B). (We will use $\mathbf{y}_{k+1}$ and τ_k instead of $\tilde{\mathbf{y}}_{k+1}$ and $\tilde{\tau}_k$ in the algorithm.)

Algorithm 2. Modified Noda iteration (MNI)

1. Given $\rho_0 \in (\rho(A,B),1), \mathbf{x}_0 > 0$ with $\|\mathbf{x}_0\| = 1$, and tol> 0.
2. **for** $k = 0,1,2,\ldots$
3. Solve $(\rho_k B - A)\mathbf{y}_{k+1} = (B-A)\mathbf{x}_k$.
4. Normalize the vector $\mathbf{x}_{k+1} = \mathbf{y}_{k+1}/\|\mathbf{y}_{k+1}\|$.
5. Compute $\rho_{k+1} = \rho_k - (1-\rho_k)\frac{\tau_k}{1-\tau_k}$, where $\tau_k = \min\frac{\mathbf{x}_k}{\mathbf{y}_{k+1}}$.

   until: $\|\rho_{k+1}B\mathbf{x}_{k+1} - A\mathbf{x}_{k+1}\| < \text{tol}$.
6. Output: $\rho(A,B) \leftarrow \rho_{k+1}$ and $\mathbf{x}_* \leftarrow \mathbf{x}_{k+1}$.

The generalized Noda iteration (GNI) presented below is based on Theorems 2.2 and 2.3. Let (A,B) be an $n\times n$ matrix pair satisfying the conditions (C1)-(C4). By Theorem 2.2 for any scalar $\rho_k \in (\rho(A,B),1]$, $\rho_k B - A$ is an invertible M-matrix and then $(\rho_k B - A)^{-1}A$ is an irreducible nonnegative matrix. Suppose $\mathbf{x}_k > 0$. The linear system

$$(\rho_k B - A)\mathbf{y}_{k+1} = A\mathbf{x}_k \tag{3.3}$$

has a unique solution $\mathbf{y}_{k+1} > 0$. Then $\mathbf{x}_{k+1} = \mathbf{y}_{k+1}/\|\mathbf{y}_{k+1}\| > 0$. Following Theorem 2.3,

$$\rho_{k+1} = \max \frac{A\mathbf{x}_{k+1}}{B\mathbf{x}_{k+1}} = \max \frac{A\mathbf{y}_{k+1}}{B\mathbf{y}_{k+1}} > \rho(A,B). \tag{3.4}$$

On the other hand, from (3.3),

$$\begin{aligned} \rho_{k+1} &= \rho_k \max \frac{A\mathbf{y}_{k+1}}{\rho_k B\mathbf{y}_{k+1}} = \rho_k \max \frac{A\mathbf{y}_{k+1}}{A(\mathbf{y}_{k+1}+\mathbf{x}_k)} \\ &= \rho_k \left[1 - \min \frac{A\mathbf{x}_k}{A(\mathbf{y}_{k+1}+\mathbf{x}_k)}\right]. \end{aligned} \tag{3.5}$$

This shows that $\rho_{k+1} \in (\rho(A,B), 1]$. Therefore, we can construct an iteration by using (3.3) to generate a positive vector sequence $\{\mathbf{x}_k\}$ and using (3.4) to generate a shift sequence $\{\rho_k\}$. The iteration process is described in the following algorithm. Note that the matrix B is not necessarily nonnegative. Although in (3.4), $B\mathbf{y}_{k+1}$ must be positive, in order to avoid of any possible numerical problems we use the formula (3.5) for generating the shifts.

Algorithm 3. Generalized Noda iteration (GNI)

1. Given $\rho_0 = 1, \mathbf{x}_0 > 0$ with $\|\mathbf{x}_0\| = 1$, and tol > 0.
2. **for** $k = 0, 1, 2, \ldots$
3. Solve $(\rho_k B - A)\mathbf{y}_{k+1} = A\mathbf{x}_k$.
4. Normalize the vector $\mathbf{x}_{k+1} = \mathbf{y}_{k+1}/\|\mathbf{y}_{k+1}\|$.
5. Compute $\rho_{k+1} = \rho_k \left[1 - \min \frac{A\mathbf{x}_k}{A\mathbf{y}_{k+1}+A\mathbf{x}_k}\right]$.

   until $\|\rho_{k+1}B\mathbf{x}_{k+1} - A\mathbf{x}_{k+1}\| < \text{tol}$.
6. Output: $\rho(A,B) \leftarrow \rho_{k+1}$ and $\mathbf{x}_* \leftarrow \mathbf{x}_{k+1}$.

The linear system in Step 3 of GNI is similar to the one in MNI. This is because if $\{\mathbf{x}_k\} \to \mathbf{x}_*$, then $A\mathbf{x}_k$ and $B\mathbf{x}_k$ are getting close to be parallel. So are $A\mathbf{x}_k$ and $(B-A)\mathbf{x}_k$. On the other hand, if $\mathbf{x}_k > 0$, it is clear that $A\mathbf{x}_k > 0$ but it is not clear whether $(B-A)\mathbf{x}_k$ is positive. Another advantage that GNI has is the choice of ρ_0. In GNI we can always simply select $\rho_0 = 1$, while in MNI ρ_0 has to be smaller than 1. Otherwise, $\rho_k = 1$ for all k and MNI fails. In fact, we can run one or more iterations of GNI to generate a shift that serves as an initial shift for MNI if necessary.

Remark 3.1. If in addition B is also nonnegative, the GNI can be modified by changing $A\mathbf{x}_k$ on the right hand side of the equation in Step 3 to $B\mathbf{x}_k$. It can be proved in the same way that the modified iteration always converges and the convergence rate is quadratic.

Remark 3.2. In view of MNI and GNI, one may consider a more general iteration formula for Step 3 as

$$(\rho_k B - A)\mathbf{y}_{k+1} = (\alpha_k A + \beta_k B)\mathbf{x}_k.$$

Note in MNI, $\alpha_k \equiv 1$ and $\beta_k \equiv -1$, and in GNI, $\alpha_k \equiv 1$ and $\beta_k \equiv 0$. How to choose α_k, β_k in each iteration to make the method more efficient? Further work needs to be done in order to answer this question.

REMARK 3.3. The matrix pair (C, D), where C is an invertible M-matrix and D is an irreducible nonnegative matrix, has a simple positive eigenvalue λ that is the smallest in modulus among all the eigenvalues. This is simply from the fact that $1/\lambda$ is the Perron root of the irreducible matirx $C^{-1}D$. So λ has a corresponding positive eigenvector. Such a matrix pair can be found in the finite element approximation of the eigenvalue problem of second-order elliptic operators ([7]). This matrix pair can be transformed to a matrix pair (A, B) that satisfies the conditions (C1)-(C4) with $A = D$ and $B = C + D$. The eigenvalue λ and the Perron root $\rho(A, B)$ have the relation $\rho(A, B) = 1/(1 + \lambda)$. Therefore, in order to compute the eigenvalue λ of (C, D), one may apply either MNI or GNI to (A, B). Certainly, both MNI and GNI can be adapted to apply to the pair (C, D) directly.

The algorithm MNI is simply modified from NI, so it always converges at a quadratic convergence rate. In the next section, we will show that GNI has exactly the same convergence behavior.

4. Convergence analysis of GNI. In this section we always assume that (A, B) is an $n \times n$ matrix pair satisfying the conditions (C1)-(C4). With this assumption from Theorem 1.1 or 2.1, (A, B) has a Perron pair $(\rho(A, B), \mathbf{x}_*)$. We will show that GNI always converges and its asymptotic convergence rate is quadratic.

Suppose $\{\mathbf{x}_k\}$ is the set of positive vectors generated by GNI. If $\mathbf{x}_k = \mathbf{x}_*$ for some k. Then from (3.4), $\rho_k = \rho(A, B)$. The next iteration breaks down, but we have obtained the exact Perron pair. In the following we always assume that $\mathbf{x}_k \neq \mathbf{x}_*$ for all k. For simplicity, we set $\rho \equiv \rho(A, B)$. Since (A, B) is regular, for any scalar μ that is not an eigenvalue of (A, B), the matrix $\mu B - A$ is nonsingular.

In the previous section we showed several properties of the sequences $\{\rho_k\}$ and $\{\mathbf{x}_k\}$ generated by GNI. We now include these properties in the following lemma.

LEMMA 4.1. *Let (A, B) be an $n \times n$ matrix pair satisfying the conditions (C1) - (C4). Let the sequences $\{\mathbf{y}_k\}, \{\mathbf{x}_k\}$, and $\{\rho_k\}$ be generated by Algorithm 3. Assume that $\mathbf{x}_k \neq \mathbf{x}_*$ for all $k \geq 0$. Then $\mathbf{x}_k > 0$ and $B\mathbf{y}_k > 0$ for all $k \geq 0$, and the shift sequence $\{\rho_k\}$ is monotonically decreasing and bounded below by ρ, i.e., $1 = \rho_0 > \rho_1 > \cdots > \rho_k > \rho_{k+1} > \cdots > \rho$.*

Proof. We prove $\mathbf{x}_k > 0$ by induction on k. From Theorem 2.2 we know that $\rho_0 B - A = B - A$ is a nonsingular M-matrix. So by Proposition 2.1, we have $(\rho_0 B - A)^{-1} \geq 0$ and the matrix $(\rho_0 B - A)^{-1}$ doesn't have a zero row. Because A is irreducible and nonnegative and $\mathbf{x}_0 > 0$,

$$\mathbf{y}_1 = (\rho_0 B - A)^{-1} A\mathbf{x}_0 > 0,$$

and then $\mathbf{x}_1 = \mathbf{y}_1/\|\mathbf{y}_1\| > 0$. This shows that $\mathbf{x}_1 > 0$.

Suppose $\mathbf{x}_k > 0$ for some $k \geq 1$. We consider the $k+1$ case. Since $\mathbf{x}_k \neq \mathbf{x}_*$, from (3.4), $\rho_k > \rho$. Similarly, we can prove $\mathbf{y}_{k+1} > 0$ and $\mathbf{x}_{k+1} > 0$.

From $\rho_k B\mathbf{y}_{k+1} = A\mathbf{x}_k + A\mathbf{y}_{k+1}$ we get $B\mathbf{y}_{k+1} > 0$ for all k. It is easy from (3.4) and Theorem 2.3 to know that the monotonicity of $\{\rho_k\}$ is from (3.4) and (3.5). ∎

LEMMA 4.2. *Let (A, B) be an $n \times n$ matrix pair satisfying the conditions (C1)-(C4), and let $\{\mathbf{x}_k\}$ be generated by Algorithm 3. Then for any convergent subsequence $\{\mathbf{x}_{k_j}\} \subseteq \{\mathbf{x}_k\}$, $\lim_{j\to\infty} \mathbf{x}_{k_j} > 0$.*

Proof. Let $\mathbf{v} = \lim_{j\to\infty} \mathbf{x}_{k_j}$. From Lemma 4.1, $\mathbf{v} \geq 0$. From (3.4) and Lemma 4.1,

$$1 = \rho_0 > \rho_{k_j} = \max \frac{A\mathbf{x}_{k_j}}{B\mathbf{x}_{k_j}} \geq \frac{(A\mathbf{x}_{k_j})^{(i)}}{(B\mathbf{x}_{k_j})^{(i)}} \to \frac{(A\mathbf{v})^{(i)}}{(B\mathbf{v})^{(i)}},$$

for all $i = 1, 2, \ldots, n$. Hence,

$$(A\mathbf{v})^{(i)} < (B\mathbf{v})^{(i)}, \quad \forall i \in \{1, 2, \ldots, n\}.$$

Suppose $\mathbf{v}^{(s)} = 0$ for some s, then the above inequality with $i = s$ implies,

$$\sum_{j=1, j\neq s}^{n} (a_{sj} - b_{sj})\mathbf{v}^{(j)} < 0,$$

which contradicts to the condition (C4). Therefore, $\mathbf{v} > 0$. ☐

LEMMA 4.3. *Let (A, B) be an $n \times n$ matrix pair satisfying the conditions (C1)-(C4), and let the positive vector sequence $\{\mathbf{y}_k\}$ be generated by Algorithm 3. Then*

$$\lim_{k\to\infty} \frac{1}{\|\mathbf{y}_k\|} = 0. \tag{4.1}$$

Proof. From (3.5), one has

$$1 - \frac{\rho_{k+1}}{\rho_k} = \min \frac{A\mathbf{x}_k}{A\mathbf{y}_{k+1} + A\mathbf{x}_k}.$$

It follows from Lemma 4.1 that $\{\rho_k\}$ converges. So

$$\lim_{k\to\infty} \left(1 - \frac{\rho_{k+1}}{\rho_k}\right) = 0.$$

Then

$$\lim_{k\to\infty} \min \frac{A\mathbf{x}_k}{A\mathbf{y}_{k+1} + A\mathbf{x}_k} = \lim_{k\to\infty} \min \frac{A\mathbf{x}_k}{\|\mathbf{y}_{k+1}\| A\mathbf{x}_{k+1} + A\mathbf{x}_k} = 0.$$

Next we show that all the components of $A\mathbf{x}_k$ are bounded below by a positive constant independent of k. If not, there exists a subsequence $\{\mathbf{x}_{k_j}\}$ such that for some integer i,

$$\lim_{j\to\infty} (A\mathbf{x}_{k_j})^{(i)} = 0. \tag{4.2}$$

Since $\|\mathbf{x}_{k_j}\| = 1$ for all k_j, we may assume that $\lim_{j\to\infty} \mathbf{x}_{k_j} = \mathbf{v}$ exists. By Lemma 4.2, $\mathbf{v} > 0$. Since A is nonnegative irreducible, we have

$$\lim_{j\to\infty} (A\mathbf{x}_{k_j})^{(i)} = \lim_{j\to\infty} (A\mathbf{v})^{(i)} > 0.$$

This is contradictory to (4.2). Therefore, $(A\mathbf{x}_k)^{(i)} \geq \beta > 0$ for some constant $\beta > 0$. Because $\|\mathbf{x}_{k_j}\| = 1$, $(A\mathbf{x}_k)^{(i)} \leq \alpha$ for any k and i for some constants $\alpha \geq \beta$. Then

$$0 = \lim_{k\to\infty} \min \frac{A\mathbf{x}_k}{\|\mathbf{y}_{k+1}\| A\mathbf{x}_{k+1} + A\mathbf{x}_k} \geq \lim_{k\to\infty} \frac{\beta}{(\|\mathbf{y}_{k+1}\| + 1)\alpha},$$

and from which one has (4.1). ◻

We also need the following results for proving the convergence properties of GNI.

Lemma 4.4. *Let (E, F) be an $n \times n$ real regular matrix pair. If a real number $\lambda \in \mathbb{R}$ and a real vector $\mathbf{x} \in \mathbb{R}^n$ satisfy*

$$E\mathbf{x} = \lambda F\mathbf{x}, \qquad \|\mathbf{x}\| = 1,$$

then there exist real nonsingular matrices U and V such that

$$E = U \begin{bmatrix} \lambda & \mathbf{e}^T \\ 0 & E_1 \end{bmatrix} V^{-1}, \quad F = U \begin{bmatrix} 1 & \mathbf{f}^T \\ 0 & F_1 \end{bmatrix} V^{-1},$$

where $U = (\gamma \mathbf{u}_1, U_2), V = (\mathbf{x}, V_2) \in \mathbb{R}^{n\times n}$ with $\gamma = \|F\mathbf{x}\| > 0$, $\mathbf{u}_1 = F\mathbf{x}/\gamma$, and $(\mathbf{u}_1, U_2), V$ are orthogonal matrices.

Proof. It is similar to the proof of Theorem 1.9 in [16, page 276]. ◻

Let (A, B) be the $n \times n$ matrix pair satisfying the conditions (C1)-(C4). Then it follows from Theorem 1.1 or Theorem 2.1 that (A, B) is a regular matrix pair and it has a Perron pair $(\rho, \mathbf{x}_*)$ satisfying

$$A\mathbf{x}_* = \rho B\mathbf{x}_*, \quad \|\mathbf{x}_*\| = 1.$$

From Lemma 4.4, there exist matrices $U = (\gamma \mathbf{u}_1, U_2), V = (\mathbf{x}_*, V_2) \in \mathbb{R}^{n\times n}$ with $\gamma = \|B\mathbf{x}_*\|$ and $\mathbf{u}_1 = B\mathbf{x}_*/\gamma$ such that $(\mathbf{u}_1, U_2)$ and V are real orthogonal, and

$$A = U \begin{bmatrix} \rho & \mathbf{a}^T \\ 0 & A_1 \end{bmatrix} V^{-1}, \quad B = U \begin{bmatrix} 1 & \mathbf{b}^T \\ 0 & B_1 \end{bmatrix} V^{-1}. \tag{4.3}$$

Let $\{\mathbf{x}_k\}$ be the unit positive vector sequence generated by Algorithm 3. For each $\mathbf{x}_k$ we express it with the (unique) combination form

$$\mathbf{x}_k = \mathbf{x}_* \cos\varphi_k + V_2 \mathbf{p}_k \sin\varphi_k, \quad \|\mathbf{p}_k\| = 1, \tag{4.4}$$

where $\varphi_k = \angle(\mathbf{x}_k, \mathbf{x}_*) = \arccos \mathbf{x}_*^T \mathbf{x}_k$. Define

$$\varepsilon_k = \rho_k - \rho > 0, \qquad C_k = \rho_k B - A.$$

Then from (4.3), one has

$$C_k = U \begin{bmatrix} \varepsilon_k & \mathbf{c}_k^T \\ 0 & L_k \end{bmatrix} V^{-1}, \qquad \mathbf{c}_k = \rho_k \mathbf{b} - \mathbf{a}, \quad L_k = \rho_k B_1 - A_1.$$

From Lemma 4.1 and Theorem 2.2, C_k is invertible. By taking the inverse on both sides of the above factorization,

$$C_k^{-1} = V \begin{bmatrix} {\varepsilon_k}^{-1} & -{\varepsilon_k}^{-1} \mathbf{c}_k^T L_k^{-1} \\ 0 & L_k^{-1} \end{bmatrix} U^{-1}. \tag{4.5}$$

The global convergence of Algorithm 3 is given in the following theorem.

Theorem 4.1. *Let (A, B) be an $n \times n$ matrix pair satisfying the conditions (C1)-(C4). Suppose that the sequences $\{\mathbf{x}_k\}$ and $\{\rho_k\}$ are generated by Algorithm 3. Then*

$$\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}_* \quad \textit{and} \quad \lim_{k\to\infty} \rho_k = \rho.$$

Moreover, the asymptotic convergence rate is quadratic.

Proof. Using the decomposition (4.4), for any k one has

$$\sin\varphi_k = \|V_2^T \mathbf{x}_k\|.$$

Using the formulas $\mathbf{x}_{k+1} = \mathbf{y}_{k+1}/\|\mathbf{y}_{k+1}\|$, $\mathbf{y}_{k+1} = C_k^{-1}A\mathbf{x}_k$, and (4.3), (4.5), one has

$$\begin{aligned}\sin\varphi_{k+1} = \|V_2^T\mathbf{x}_{k+1}\| &= \frac{\|V_2^TC_k^{-1}A\mathbf{x}_k\|}{\|\mathbf{y}_{k+1}\|} = \frac{\|L_k^{-1}A_1V_2^T\mathbf{x}_k\|}{\|\mathbf{y}_{k+1}\|}\\ &\le \frac{\|L_k^{-1}\|\|A_1\|\|V_2^T\mathbf{x}_k\|}{\|\mathbf{y}_{k+1}\|} = \frac{\|L_k^{-1}\|\|A_1\|}{\|\mathbf{y}_{k+1}\|}\sin\varphi_k. \end{aligned} \tag{4.6}$$

By Theorem 2.1, ρ is a simple eigenvalue of (A, B) and (A, B) does not have an eigenvalue on $(\rho, 1]$. So (A_1, B_1) does not have any eigenvalues on the closed interval $[\rho, 1]$. Since $\rho_k \in (\rho, 1]$, for any k,

$$\|L_k^{-1}\| = \|(\rho_kB_1 - A_1)^{-1}\| \le \max_{t\in[\rho,1]} \|(tB_1 - A_1)^{-1}\|,$$

i.e., $\|L_k^{-1}\|$ is uniformly bounded above by a positive constant. Now, from (4.1), and (4.6) one has $\lim\limits_{k\to\infty} \sin\varphi_k = 0$, or equivalently $\lim\limits_{k\to\infty} \mathbf{x}_k = \mathbf{x}_*$. Then,

$$\lim_{k\to\infty}\rho_k = \lim_{k\to\infty}\max\frac{A\mathbf{x}_k}{B\mathbf{x}_k} = \max\left(\lim_{k\to\infty}\frac{A\mathbf{x}_k}{B\mathbf{x}_k}\right) = \max\frac{A\mathbf{x}_*}{B\mathbf{x}_*} = \rho.$$

For proving the convergence rate, we first relate ε_k to ρ_k. Using (4.4),

$$\begin{aligned}\varepsilon_k = \rho_k - \rho = \max\frac{A\mathbf{x}_k}{B\mathbf{x}_k} - \rho &= \max\frac{(A-\rho B)\mathbf{x}_k}{B\mathbf{x}_k}\\ = \sin\varphi_k \max\frac{(A-\rho B)V_2\mathbf{p}_k}{B\mathbf{x}_k} &=: \tau_k\sin\varphi_k.\end{aligned}$$

Then, following (4.3), (4.4), and (4.5),

$$\begin{aligned}\mathbf{y}_{k+1} = C_k^{-1}A\mathbf{x}_k &= V\begin{bmatrix}\varepsilon_k^{-1}\rho & \varepsilon_k^{-1}(\mathbf{a}^T - \mathbf{c}_k^TL_k^{-1}A_1)\\ 0 & L_k^{-1}A_1\end{bmatrix}V^{-1}\mathbf{x}_k\\ &= \varepsilon_k^{-1}(\rho\cos\varphi_k + (\mathbf{a}^T - \mathbf{c}_k^TL_k^{-1}A_1)\mathbf{p}_k\sin\varphi_k)\mathbf{x}_* + V_2L_k^{-1}A_1\mathbf{p}_k\sin\varphi_k,\end{aligned}$$

and

$$\mathbf{x}_{k+1} = \frac{(\rho\cos\varphi_k + (\mathbf{a}^T - \mathbf{c}_k^TL_k^{-1}A_1)\mathbf{p}_k\sin\varphi_k)\mathbf{x}_* + V_2L_k^{-1}A_1\mathbf{p}_k\varepsilon_k\sin\varphi_k}{\|(\rho\cos\varphi_k + (\mathbf{a}^T - \mathbf{c}_k^TL_k^{-1}A_1)\mathbf{p}_k\sin\varphi_k)\mathbf{x}_* + V_2L_k^{-1}A_1\mathbf{p}_k\varepsilon_k\sin\varphi_k\|}.$$

Using the formula in (4.6) again,

$$\begin{aligned}\sin\varphi_{k+1} &= \|V_2^T\mathbf{x}_{k+1}\|\\ &= \frac{\|L_k^{-1}A_1\mathbf{p}_k\|\varepsilon_k\sin\varphi_k}{\|(\rho\cos\varphi_k + (\mathbf{a}^T - \mathbf{c}_k^TL_k^{-1}A_1)\mathbf{p}_k\sin\varphi_k)\mathbf{x}_* + V_2L_k^{-1}A_1\mathbf{p}_k\varepsilon_k\sin\varphi_k\|}.\end{aligned}$$

Since $\varphi_k \to 0$, one has $\sin\varphi_k \to 0$ and $\cos\varphi_k = 1 + O(\sin^2\varphi_k)$. Because $\|L_k^{-1}\|$ is uniformly bounded from above and $\mathbf{p}_k$ is a unit vector, from the above relation

$$\sin\varphi_{k+1} = \frac{\|L_k^{-1}A_1\mathbf{p}_k\|\tau_k}{\rho}\sin^2\varphi_k + O(\sin^3\varphi_k).$$

Note τ_k is positive (because ε_k and $\sin\varphi_k$ are), and because $B\mathbf{x}_k \to B\mathbf{x}_*$ and $\mathbf{p}_k$ is a unit vector, τ_k is bounded from above as well. The above relation shows the quadratic convergence rate of GNI. ◻

5. Numerical experiments. In this section we present numerical experiments to compare NI, MNI, GNI and shift-invert power method (SIP)[1]. The SIP that we used is just the Rayleigh quotient iteration. In these examples, the initial vector is selected to be $x_0 = \frac{1}{\sqrt{n}}[1,\dots,1]^T \in R^n$. The initial eigenvalue is $\lambda_0 = \max\frac{(B-A)^{-1}A\mathbf{x}_0}{\mathbf{x}_0}$ for NI, and $\rho_0 = \frac{\lambda_0}{1+\lambda_0}$ for MNI, GNI and SIP. All the computations are performed using Matlab, version 6.5. with the machine precision about 2.22×10^{-16}.

Example 5.1. [2, example 3.7] Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2(1+\varepsilon^2) & 3-\varepsilon^2 & 0 \end{bmatrix}$$

with $3 > \varepsilon^2$, and $B = I + A$ be another nonnegative matrix. For the pair (A, B) the conditions (C1)-(C4) are satisfied and the matrix $B - A = I$ is well-conditioned. The Perron root of (A, B) is $\rho = \rho(A, B) = \frac{2}{3}$, and the other eigenvalues are $1 \pm \frac{i}{\varepsilon}$. Taking $\varepsilon = 1.7$, Figure 1 depicts how the residual norm evolves versus the number of iterations for the NI, MNI, GNI and SIP, respectively. Note that the NI, MNI and GNI use 7 iterations to achieve the machine precision, and SIP only uses 5 iterations. However, SIP does not preserve the positivity of the vectors during the iteration process. In this example, the vectors computed in the second, third and fifth iterations of SIP are actually negative, i.e., $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_5 < 0$. In this example, MNI and GNI are comparable.

Example 5.2. Let

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and

$$B - A = 6.00001 \times I - N, \quad N = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 1 & 5 \end{bmatrix},$$

where B has negative entries. Since $\rho(N) = 6$, the matrix $B - A$ is a nonsingular M-matrix and is not well-conditioned. In fact,

$$\mathbf{cond}(B - A) = \|B - A\|\|(B - A)^{-1}\| = 6.831441069381088e + 005.$$

Figure 2 depicts the residual norms of NI, MNI, GNI and SIP, respectively. Note that both GNI and SIP use 2 and 3 iterations to achieve the machine precision, respectively. However both NI and MNI uses 5 iterations to achieve the machine precision. NI,

Figure 1: Residual norms of NI, MNI, GNI and SIP

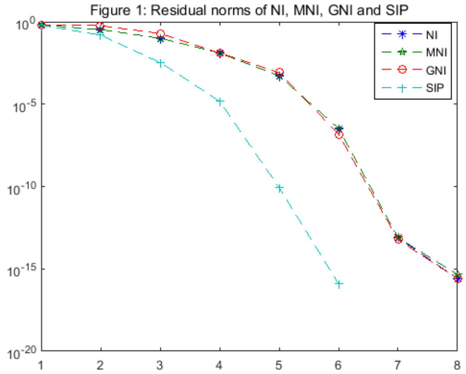
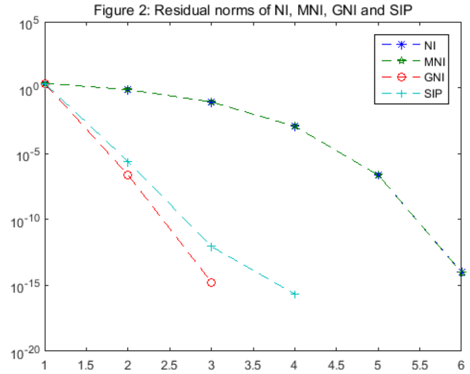


Figure 2: Residual norms of NI, MNI, GNI and SIP



MNI and GNI always preserve the positivity of the vectors. For SIP, it again produces negative vectors, which are $\mathbf{x}_2$ and $\mathbf{x}_3$. In this example GNI outperforms MNI.

Example 5.3. [7] Consider the second-order elliptic operator

$$\left\{ \begin{array}{lcll} Lu & \equiv & -\Delta \cdot (\mathbb{D}\Delta u) + \mathbf{b} \cdot \Delta u + cu = \lambda u, & \text{in } \Omega, \\ u & = & 0, & \text{on } \partial\Omega, \end{array} \right. \tag{5.1}$$

where $\Omega \subset \mathbb{R}^d (d \geq 1)$ is a polyhedron, $\mathbb{D} = \mathbb{D}(x) : \Omega \to \mathbb{R}^{d\times d}$ is symmetric positive definite for all $x \in \Omega$, $\mathbf{b} = \mathbf{b}(x) : \Omega \to \mathbb{R}^d$ and $c = c(x) : \Omega \to \mathbb{R}$ are sufficiently smooth satisfying $c(x) - \frac{1}{2}\nabla \cdot b(x) \geq 0$ for all $x \in \Omega$. Under these conditions, the operator L has a simple *principal* eigenvalue λ_1 and a corresponding positive eigenfunction $0 < u_1(x) \in H_0^1(\Omega)$, in the sense that for any eigenvalue λ of L one has $\mathrm{Re}(\lambda) \geq \lambda_1$. Using a P1 finite-element discretization the above eigenvalue problem is approximated by the eigenvalue problem of a matrix pair (C, D), i.e.,

$$C\mathbf{x} = \lambda D\mathbf{x}, \tag{5.2}$$

where C is the stiffness matrix and D is the mass matrix. In [7], it gives conditions on the meshes so that in the resulting matrix pair, C is an invertible M-matrix and D is irreducible nonnegative. Such a matrix (C, D) has a simple positive principal eigenvalue $\lambda_{\min}$ with a corresponding positive eigenvector. This is clear, since $1/\lambda_{\min}$ is the Perron root of the irreducible nonnegative matrix $C^{-1}D$. In this way, $\lambda_{\min}$ not just approximates λ_1 numerically, it inherits the physical feature of λ_1 as well.

In this example we test our iterations for computing $\lambda_{\min}$ and its unit positive eigenvector of the matrix pair (C, D). The pair is generated by setting $d = 2$, $\Omega = [0, 1] \times [0, 1]$, $\mathbb{D} = \begin{bmatrix} 10 & 9 \\ 9 & 0 \end{bmatrix}$, $\mathbf{b}(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $c(x) = 0$. The mesh is generated by first applying a rectangular mesh that cuts the unit square Ω into $(m-1)^2$ equally sized small squares, then by cutting each small square into two right triangles by lining up the northeast and southwest vertices. As a result the size of the matrices C, D is $(m-2)^2 \times (m-2)^2$. We us $m = 8, 10, 12, 16$ to generate 4 pairs of (C, D) with size $36 \times 36, 64 \times 64, 100 \times 100$, and 196×196, respeictively. Based on Remark 3.3, with each size we apply the iterations to the pair $(D, C + D)$ instead to compute its Perron root ρ, then $\lambda_{\min} = 1/\rho - 1$. Figure 3, 4, 5, and 6, respectively, give the relation between residual norm $\|(C - \lambda_k D)\mathbf{x}_k\|$ (with $\lambda_k = 1/\rho_k - 1$) and the number of iterations for each of NI, MNI, GNI and SIP for different sizes. The results show that the GNI always converges faster than the others. Again, SIP don't preserve the positivity of the vectors generated during iteration process.

Figure 3: Residual norms of NI, MNI, GNI and SIP
for the 36*36 matrix pair (C, D)

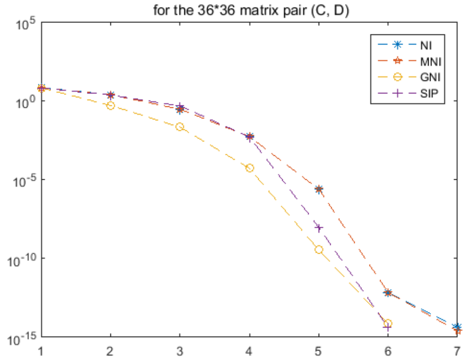


Figure 4: Residual norms of NI, MNI, GNI and SIP
for the 64*64 matrix pair (C, D)

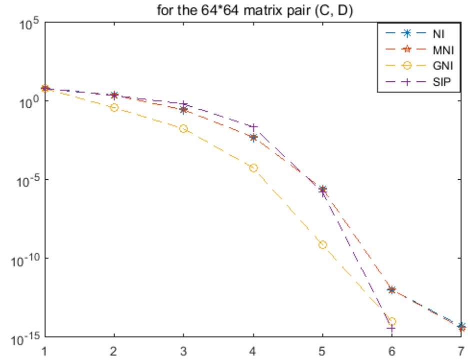


Figure 5: Residual norms of NI, MNI, GNI and SIP
for the 100*100 matrix pair (C, D)

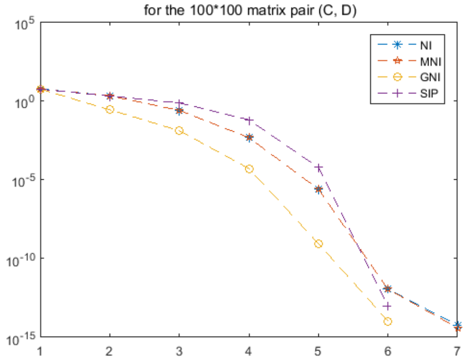
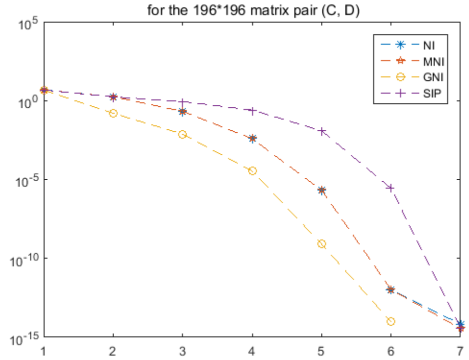


Figure 6: Residual norms of NI, MNI, GNI and SIP
for the 196*196 matrix pair (C, D)



6. Conclusions. In this paper, the modified Noda iteration (MNI) and generalized Noda iteration (GNI) have been developed for computing the Perron pair of matrix pairs that follow the Perron-Frobenius theory. By using the techniques similar to that in [8, 9] we proved that GNI is always convergent and it converges quadratically. MNI has a similar convergence behavior, but GNI is slightly simpler and Numerical examples show that GNI performs slightly faster. It needs further study for comparing the actual convergence behaviors of GNI and MNI. Another work needs to be done is to develop an iterative method as the one in [18] so that the Perron pair can be computed with high accuracy.

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