Robust formulas for optimal H_∞ controllers

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Abstract

We present formulas for the construction of optimal H_∞ controllers that can be implemented in a numerically robust way. We base the formulas on the γ-iteration developed in [6]. The controller formulas proposed here avoid the solution of algebraic Riccati equations with their problematic matrix inverses and matrix products. They are also applicable in the neighborhood of the optimal γ, where the classical formulas may call for the inverse of singular or ill-conditioned matrices. The advantages of the new formulas are demonstrated by several numerical examples.

Key words: H_∞ control; controller design; optimal controller; CS decomposition; Lagrangian subspaces; even pencil.

1 Introduction

The optimal infinite-horizon output (or measurement) feedback H_∞ control problem is one of the central tasks in robust control, see, e.g., [13,18,21,22], but the development of robust numerical methods for the H_∞ control is unusually difficult [20]. The classic γ-iteration often used in optimal H_∞ control computations encounters several finite precision arithmetic hazards that often limit its accuracy as a numerical method. A new numerical method for the γ-iteration suggested in [6] has significantly better robustness in the presence of rounding errors. Based on this approach, this paper proposes a numerical method for the implementation of the associated optimal controllers. Note that another variant of controller formulas based on the the γ-iteration from [6] is suggested in [15]. Our approach differs in the derivation and form of the controller formulas. Moreover, we have implemented our formulas and we will present numerical results obtained with these formulas.

Consider the linear control system

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \qquad x(t_0) = x^0, \\ z &= C_1 x + D_{11} w + D_{12} u, \\ y &= C_2 x + D_{21} w + D_{22} u, \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m_i}$, $C_i \in \mathbb{R}^{p_i\times n}$, and $D_{ij} \in \mathbb{R}^{p_i\times m_j}$ for $i,j = 1,2$. (By $\mathbb{R}^{n\times k}$ we denote the set of real $n \times k$ matrices.) As usual, see [13,22], we assume $p_1 \geq m_2$ and $m_1 \geq p_2$. In this system, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{m_2}$ is the control input vector, and $w(t) \in \mathbb{R}^{m_1}$ is an exogenous input that may include noise, linearization errors and unmodeled dynam-

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ics. The vector $y(t) \in \mathbb{R}^{p_2}$ contains measured outputs, while $z(t) \in \mathbb{R}^{p_1}$ is a regulated output or error.

The optimal H_∞ control problem: Determine a dynamic controller

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}_1 y + \hat{B}_2 \hat{u}, \\ u &= \hat{C}_1 \hat{x} + \hat{D}_{11} y + \hat{D}_{12} \hat{u}, \\ \hat{y} &= \hat{C}_2 \hat{x} + \hat{D}_{21} y, \end{aligned} \tag{2}$$

with $\hat{A} \in \mathbb{R}^{N\times N}$, $\hat{B}_1 \in \mathbb{R}^{N\times p_2}$, $\hat{B}_2 \in \mathbb{R}^{N\times q_1}$, $\hat{C}_1 \in \mathbb{R}^{m_2\times N}$, $\hat{C}_2 \in \mathbb{R}^{q_2\times N}$, $\hat{D}_{11} \in \mathbb{R}^{m_2\times p_2}$, $\hat{D}_{12} \in \mathbb{R}^{m_2\times q_1}$, $\hat{D}_{21} \in \mathbb{R}^{q_2\times p_2}$ such that the closed-loop system resulting from the combined system of (1) and (2),

(1) is *internally stable*, i.e., the solution of the system with $w \equiv 0$ is asymptotically stable, and
(2) the closed-loop transfer function T_{zw} from w to z is minimized in the H_∞ norm.

For a matrix valued rational function $F(s)$ that is analytic in the open right-half plane, the H_∞ norm is given by $\|F\|_\infty = \sup_{\omega\in\mathbb{R}} \sigma_{\max}[F(\imath\omega)]$, where $\sigma_{\max}[F(\imath\omega)]$ denotes the maximal singular value of the matrix $F(\imath\omega)$. If $F(s)$ is the transfer function of a control system with noise or disturbance inputs, then $\|F\|_\infty$ is a measure of the worst case influence of the disturbances on the output. The solution of this problem is usually approached via the modified optimal H_∞ control problem:

The modified optimal H_∞ control problem: Let Γ be the set of numbers $\gamma > 0$ for which there exists an internally stabilizing dynamic controller (2) such that the closed loop transfer function T_{zw} satisfies $\gamma > \|T_{zw}\|_\infty$. Determine $\gamma_{mo} = \inf \Gamma$.

Because there may be no dynamic controller that leads to a transfer function that actually achieves H_∞ norm equal to γ_{mo}, in general, one must use a controller whose transfer function has larger H_∞ norm, i.e., an internally stabilizing dynamic controller such that the closed loop transfer function satisfies $\|T_{zw}\|_\infty < \gamma$ for some $\gamma > \gamma_{mo}$. Such a controller is usually called a *suboptimal controller*. The *γ-iteration* is the iterative root finding process of determining an approximation to γ_{mo}. Classical numerical methods for determining γ_{mo} are based on the solution of Riccati equations or Lagrangian invariant subspaces, see [11,13,14,18,22] and are implemented in software packages like MATLAB® or SLICOT, [7–9]. A more robust method for carrying out the γ-iteration has recently been proposed in [6].

Once a sufficiently accurate approximation to γ_{mo} is determined, a suboptimal controller can be constructed using the mathematically correct, but numerically hazardous formulas suggested in [14,22] which we recall in Section 2, or by the more robust formulas that we present in Section 3. We will demonstrate the quality of the new formulas with several numerical examples in Section 4 and give some final remarks in Section 5.

2 Preliminaries

The formulas for designing optimal controllers are quite technical and only hold under some suitable assumptions. In this section we review the classical formulas and the assumptions under which H_∞ norm calculations typically operate.

A typical set of assumptions for the solution of the modified optimal H_∞ control is as follows [13,14,18,22]:

A1. The pair (A, B_2) is *stabilizable* and the pair (A, C_2) is *detectable*, i.e., $\operatorname{rank}[A - \lambda I, B_2] = \operatorname{rank}[A^T - \lambda I, C_2^T] = n$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$.
A2. $D_{22} = 0$ and both D_{12} and D_{21} have full rank.
A3. The matrix $\begin{bmatrix} A-\imath\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all real ω.
A4. The matrix $\begin{bmatrix} A-\imath\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all real ω.

Remark 2.1 The requirement that $D_{22} = 0$ (Assumption A2) is for convenience. It is not a fundamental restriction, since systems that have a direct link from input to output, i.e., for which $D_{22} \neq 0$, can be synthesized by first studying the problem without this term, see [22].

Following the notation in [22], we introduce the following two symmetric matrices formed from the matrices D_{ij} and a parameter $\gamma \in \mathbb{R}$,

$$\begin{aligned} R_H(\gamma) &:= \begin{bmatrix} D_{11}^T \\ D_{12}^T \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \end{bmatrix} - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}, \\ R_J(\gamma) &:= \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} \begin{bmatrix} D_{11}^T & D_{21}^T \end{bmatrix} - \begin{bmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{3}$$

These matrices play an essential role in the theory of optimal H_∞ control problems, see [14,22], and the classical numerical methods require both $R_H(\gamma)$ and $R_J(\gamma)$ to be nonsingular. Under Assumption A2, there exist only a finite number of nonnegative values γ for which (at least) one of the matrices $R_H(\gamma)$ and $R_J(\gamma)$ is singular. Let $\hat{\gamma}$ be the largest γ value for which this is the case. If $D_{11} = 0$, then $\hat{\gamma} = 0$; otherwise, $\hat{\gamma}$ is typically positive. Note that by definition, $\gamma_{mo} > \hat{\gamma}$.

Let

$$D_{12} = U_{12} \begin{bmatrix} 0 \\ \Sigma_{12} \end{bmatrix} V_{12}^T, \quad D_{21} = V_{21} \begin{bmatrix} 0 & \Sigma_{21} \end{bmatrix} U_{21}^T, \tag{4}$$

be (slightly permuted) singular value decompositions (see [12]) of D_{12} and D_{21} with real orthogonal matrices U_{12}, U_{21}, V_{12}, V_{21} and nonnegative diagonal matrices Σ_{12}, Σ_{21}. The diagonal entries of Σ_{12} and Σ_{21} are the singular values of D_{12} and D_{21}, respectively. Then define $\bar{D}_{11}$, $\bar{D}_{12}$, and $\bar{D}_{21}$ in terms of D_{11}, D_{12}, D_{21} and (4) by

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{bmatrix} = \begin{bmatrix} U_{12} & 0 \\ 0 & V_{21}\Sigma_{21} \end{bmatrix} \begin{bmatrix} \bar{D}_{11} & \bar{D}_{12} \\ \bar{D}_{21} & 0 \end{bmatrix} \begin{bmatrix} U_{21}^T & 0 \\ 0 & \Sigma_{12}V_{12}^T \end{bmatrix}. \tag{5}$$

Note that by assumption $D_{22} = 0$ and the described transformation does not change this. It follows from (4) that $\bar{D}_{21} = [0,\, I_{p_2}]$ and $\bar{D}_{12} = \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}$. This induces a finer partition of $\bar{D}_{11}$ so that

$$\begin{bmatrix} \bar{D}_{11} & \bar{D}_{12} \\ \bar{D}_{21} & 0 \end{bmatrix} = \begin{bmatrix} D_1 & D_2 & 0 \\ D_3 & D_4 & I_{m_2} \\ 0 & I_{p_2} & 0 \end{bmatrix}. \tag{6}$$

Now, under Assumption A2, for $\hat{\gamma}$ as defined above, we have

$$\hat{\gamma} = \max\left(\sigma_{\max}\begin{bmatrix} D_1 & D_2 \end{bmatrix},\, \sigma_{\max}\begin{bmatrix} D_1 \\ D_3 \end{bmatrix}\right),$$

where $\sigma_{\max}(M)$ denotes the maximal singular value of the matrix M.

The classical approach for the computation of γ_{mo}, see, e.g., [21,22], employs the solution of algebraic Riccati equations (AREs). Consider the Hamiltonian matrices

$$\begin{aligned} H(\gamma) &= \begin{bmatrix} H_1(\gamma) & H_2(\gamma) \\ H_3(\gamma) & -H_1(\gamma)^T \end{bmatrix} \\ &= \begin{bmatrix} A & 0 \\ -C_1^T C_1 & -A^T \end{bmatrix} \\ &\quad - \begin{bmatrix} B_1 & B_2 \\ -C_1^T D_{11} & -C_1^T D_{12} \end{bmatrix} R_H^{-1}(\gamma) \begin{bmatrix} D_{11}^T C_1 & B_1^T \\ D_{12}^T C_1 & B_2^T \end{bmatrix}, \end{aligned} \tag{7}$$

$$\begin{aligned} J(\gamma) &= \begin{bmatrix} J_1(\gamma) & J_2(\gamma) \\ J_3(\gamma) & -J_1(\gamma)^T \end{bmatrix} \\ &= \begin{bmatrix} A^T & 0 \\ -B_1 B_1^T & -A \end{bmatrix} \\ &\quad - \begin{bmatrix} C_1^T & C_2^T \\ -B_1 D_{11}^T & -B_1 D_{21}^T \end{bmatrix} R_J^{-1}(\gamma) \begin{bmatrix} D_{11} B_1^T & C_1 \\ D_{21} B_1^T & C_2 \end{bmatrix}, \end{aligned} \tag{8}$$

and the associated γ-dependent AREs

$$\begin{aligned} &H_1(\gamma)X_H(\gamma) + X_H(\gamma)H_1(\gamma)^T \\ &+X_H(\gamma)H_2(\gamma)X_H(\gamma) - H_3(\gamma) = 0, \end{aligned} \tag{9}$$

and

$$\begin{aligned} &J_1(\gamma)X_J(\gamma) + X_J(\gamma)J_1(\gamma)^T \\ &+X_J(\gamma)J_2(\gamma)X_J(\gamma) - J_3(\gamma) = 0. \end{aligned} \tag{10}$$

Classically, one computes the unique symmetric positive semidefinite (stabilizing) symmetric solutions $X_H(\gamma)$ and $X_J(\gamma)$ of (9), (10), respectively, or what is more numerically stable, invariant subspaces of the associated Hamiltonian matrices, see [22, Ch. 16–17]. The latter approach determines symmetric matrices X_H, X_J matrices such that

$$H(\gamma)\begin{bmatrix} I_n \\ X_H \end{bmatrix} = \begin{bmatrix} I_n \\ X_H \end{bmatrix} T_H, \quad J(\gamma)\begin{bmatrix} I_n \\ X_J \end{bmatrix} = \begin{bmatrix} I_n \\ X_J \end{bmatrix} T_J,$$

for some $n \times n$ matrices T_H and T_J, respectively, with all their eigenvalues in the open left half complex plane.

Remark 2.2 The columns of the matrices $\begin{bmatrix} I_n \\ X_H \end{bmatrix}$ and $\begin{bmatrix} I_n \\ X_J \end{bmatrix}$ form unique *Lagrangian invariant subspaces*. Under some further assumptions, see [10,19], such unique Lagrangian invariant subspaces still exist, even when eigenvalues are on the imaginary axis.

In terms of X_H, X_J (we leave off the dependency on γ in the following) and the original data we then define the matrices

$$F = -R_H^{-1}\left(\begin{bmatrix} D_{11}^T \\ D_{12}^T \end{bmatrix} C_1 + \begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} X_H\right) =: \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \tag{11a}$$

$$\begin{aligned} L &= -\left(B_1\begin{bmatrix} D_{11}^T & D_{21}^T \end{bmatrix} + X_J\begin{bmatrix} C_1^T & C_2^T \end{bmatrix}\right) R_J^{-1} \\ &=: \begin{bmatrix} L_1 & L_2 \end{bmatrix}, \end{aligned} \tag{11b}$$

$$Z = \left(I_n - \gamma^{-2} X_J X_H\right)^{-1}, \tag{11c}$$

where R_H and R_J are defined in (3). Once γ_{mo}, the optimal value of γ, has been determined, then for all $\gamma > \gamma_{mo}$, one has (see [6,22]) that $R_H(\gamma)$, $R_J(\gamma)$ are nonsingular; the matrices X_H, and X_J exist; and $\gamma^2 > \rho(X_J X_H)$, where $\rho(X_J X_H)$ is the spectral radius of $X_J X_H$. Therefore, for every $\gamma > \gamma_{mo}$, the matrices F, L, Z are well defined.

Then, for a given number $\gamma \geq \gamma_{mo}$, a suboptimal controller (2) is usually constructed by using the following formulas ([22, Theorem 17.1]).

$$
\begin{aligned}
(a)\quad & \hat{D}_{11} = -V_{12}\Sigma_{12}^{-1}\cdot \\
& \left(D_3 D_1^T \left(\gamma^2 I - D_1 D_1^T\right)^{-1} D_2 + D_4\right) \Sigma_{21}^{-1} V_{21}^T, \\
(b)\quad & \hat{D}_{12}\hat{D}_{12}^T = V_{12}\Sigma_{12}^{-1}\cdot \\
& \left(I_{m_2} - D_3 \left(\gamma^2 I - D_1^T D_1\right)^{-1} D_3^T\right) \Sigma_{12}^{-1} V_{12}^T, \\
(c)\quad & \hat{D}_{21}^T\hat{D}_{21} = V_{21}\Sigma_{21}^{-1}\cdot \\
& \left(I_{p_2} - D_2^T \left(\gamma^2 I - D_1 D_1^T\right)^{-1} D_2\right) \Sigma_{21}^{-1} V_{21}^T, \\
(d)\quad & \hat{B}_2 = Z(B_2 + L_1 D_{12})\hat{D}_{12}, \\
(e)\quad & \hat{B}_1 = Z[(B_2 + L_1 D_{12})\hat{D}_{11} - L_2], \\
(f)\quad & \hat{C}_2 = -\hat{D}_{21}(C_2 + D_{21}F_1), \\
(g)\quad & \hat{C}_1 = F_2 - \hat{D}_{11}(C_2 + D_{21}F_1), \\
(h)\quad & \hat{A} = A + \begin{bmatrix} B_1 & B_2 \end{bmatrix} F - \hat{B}_1(C_2 + D_{21}F_1).
\end{aligned}
\tag{12}
$$

The basis for the robust method derived in [6] to compute γ_{mo} is to avoid all inversions, matrix products and sums, as well as solutions to AREs, that are used in the classical γ-iteration. For this, the eigenvalue problems for the Hamiltonian matrices $H(\gamma)$ and $J(\gamma)$ are first replaced by generalized eigenvalue problems for the following two *even* (skew-symmetric/symmetric) matrix pencils that only contain original data from (1):

$\lambda N - M_H(\gamma) :=$

$$
\lambda \left[\begin{array}{cc|ccc} 0 & I_n & 0 & 0 & 0 \\ -I_n & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{cc|ccc} 0 & -A^T & 0 & 0 & -C_1^T \\ -A & 0 & B_1 & B_2 & 0 \\ \hline 0 & B_1^T & \gamma^2 I_{m_1} & 0 & D_{11}^T \\ 0 & B_2^T & 0 & 0 & D_{12}^T \\ -C_1 & 0 & D_{11} & D_{12} & I_{p_1} \end{array}\right],
$$

$\lambda N - M_J(\gamma) :=$

$$
\lambda \left[\begin{array}{cc|ccc} 0 & I_n & 0 & 0 & 0 \\ -I_n & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{cc|ccc} 0 & -A & 0 & 0 & -B_1 \\ -A^T & 0 & C_1^T & C_2^T & 0 \\ \hline 0 & C_1 & \gamma^2 I_{p_1} & 0 & D_{11} \\ 0 & C_2 & 0 & 0 & D_{21} \\ -B_1^T & 0 & D_{11}^T & D_{21}^T & I_{m_1} \end{array}\right].
$$

Denote by $\hat{\gamma}^I$ the largest γ value for which at least one of these pencils has a purely imaginary eigenvalue. It has been shown in [6] that if the assumptions A1–A4 are satisfied, then for all $\gamma > \hat{\gamma}^I$, the pencils $\lambda N - M_H(\gamma)$ and $\lambda N - M_J(\gamma)$ each have a unique n-dimensional deflating subspace corresponding to the eigenvalues in the open left half plane and for $\gamma \to \hat{\gamma}^I$ there still exists a unique n-dimensional deflating subspace corresponding to the eigenvalues in the closed left half plane.

If the columns of the matrices

$$
Q_H = \begin{array}{c} \\ n \\ n \\ m_1 \\ m_2 \\ p_1 \end{array} \begin{array}{c} n \\ \begin{bmatrix} Q_{H,1} \\ Q_{H,2} \\ Q_{H,3} \\ Q_{H,4} \\ Q_{H,5} \end{bmatrix} \end{array}, \qquad Q_J = \begin{array}{c} \\ n \\ n \\ p_1 \\ p_2 \\ m_1 \end{array} \begin{array}{c} n \\ \begin{bmatrix} Q_{J,1} \\ Q_{J,2} \\ Q_{J,3} \\ Q_{J,4} \\ Q_{J,5} \end{bmatrix} \end{array}
$$

span these unique deflating subspaces, then the columns of the submatrices

$$
\begin{bmatrix} Q_{H,1} \\ Q_{H,2} \end{bmatrix}, \qquad \begin{bmatrix} Q_{J,1} \\ Q_{J,2} \end{bmatrix}
$$

span the desired Lagrangian invariant subspaces of the Hamiltonian matrices $H(\gamma)$ in (7) and $J(\gamma)$ in (8), respectively, i.e., $Q_{H,1}^T Q_{H,2} = Q_{H,2}^T Q_{H,1}$ and $Q_{J,1}^T Q_{J,2} = Q_{J,2}^T Q_{J,1}$, see [6,10,19]. Furthermore, the symmetric positive semidefinite (stabilizing) solutions of the AREs (9) and (10), if they exist, can be expressed as

$$
X_H = Q_{H,2} Q_{H,1}^{-1}, \qquad X_J = Q_{J,2} Q_{J,1}^{-1}.
$$

In order to avoid explicitly forming X_H, X_J, in [6] the following technique was introduced. Let the columns of

$$
\begin{bmatrix} X_{H,1} \\ X_{H,2} \end{bmatrix}, \quad \begin{bmatrix} X_{J,1} \\ X_{J,2} \end{bmatrix} \tag{13}
$$

form orthonormal bases of the Lagrangian invariant subspaces $\left[\begin{smallmatrix} Q_{H,1} \\ Q_{H,2} \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} Q_{J,1} \\ Q_{J,2} \end{smallmatrix}\right]$, respectively. These may be determined by QR factorization or the modified Gram-Schmidt process [12]. Note again that we have dropped the explicit γ dependency in $X_{H,i}, X_{J,i}$, $i = 1, 2$.

Introduce the symmetric matrix

$$
\begin{aligned}
\mathcal{Y}(\gamma) &:= \begin{bmatrix} \gamma X_{H,2}^T X_{H,1} & X_{H,2}^T X_{J,2} \\ X_{J,2}^T X_{H,2} & \gamma X_{J,2}^T X_{J,1} \end{bmatrix} \\
&= \begin{bmatrix} X_{H,2}^T & 0 \\ 0 & X_{J,2}^T \end{bmatrix} \begin{bmatrix} \gamma X_{H,1} & X_{J,2} \\ X_{H,2} & \gamma X_{J,1} \end{bmatrix}.
\end{aligned}
$$

If $\gamma \leq \hat{\gamma}^I$, then the pencils $\lambda N - M_H(\gamma)$ and $\lambda N - M_J(\gamma)$ may or may not have a unique n-dimensional deflating subspace corresponding to the eigenvalues in the closed left-half plane. If for a particular $\gamma \leq \hat{\gamma}^I$, such unique deflating subspaces do not exist, then $\mathcal{Y}(\gamma)$ is not defined.

The optimal value γ_{mo} is then determined in [6] and [22, Theorem 16.16] by detecting a rank change in the matrix $\mathcal{Y}(\gamma)$. For this and for the formulas in Section 3 we need the following well-known lemma on the *CS decomposition.*

Lemma 2.3 [17] *If X_1, $X_2 \in \mathbb{R}^{n,n}$ and the columns of $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ form an orthonormal basis of a Lagrangian subspace, i.e., $X_1^T X_2 = X_2^T X_1$, then there exist orthogonal matrices $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{n\times n}$ such that $U^T X_1 V = C$ and $U^T X_2 V = S$ are both diagonal and $C^2 + S^2 = I$.*

If for a given value of γ we apply Lemma 2.3 to $\begin{bmatrix} X_{H,1} \\ X_{H,2} \end{bmatrix}$, $\begin{bmatrix} X_{J,1} \\ X_{J,2} \end{bmatrix}$, then we get

$$U_H^T X_{H,1} V_H = C_H =: \begin{array}{c} r_H \\ k_H \\ n-t_H \end{array} \overset{\begin{array}{ccc} r_H & k_H & n-t_H \end{array}}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & \Sigma_H & 0 \\ 0 & 0 & I \end{bmatrix}}, \tag{14a}$$

$$U_H^T X_{H,2} V_H = S_H =: \begin{array}{c} r_H \\ k_H \\ n-t_H \end{array} \overset{\begin{array}{ccc} r_H & k_H & n-t_H \end{array}}{\begin{bmatrix} I & 0 & 0 \\ 0 & \Delta_H & 0 \\ 0 & 0 & 0 \end{bmatrix}}, \tag{14b}$$

$$U_J^T X_{J,1} V_J = C_J =: \begin{array}{c} r_J \\ k_J \\ n-t_J \end{array} \overset{\begin{array}{ccc} r_J & k_J & n-t_J \end{array}}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & \Sigma_J & 0 \\ 0 & 0 & I \end{bmatrix}}, \tag{14c}$$

$$U_J^T X_{J,2} V_J = S_J =: \begin{array}{c} r_J \\ k_J \\ n-t_J \end{array} \overset{\begin{array}{ccc} r_J & k_J & n-t_J \end{array}}{\begin{bmatrix} I & 0 & 0 \\ 0 & \Delta_J & 0 \\ 0 & 0 & 0 \end{bmatrix}}, \tag{14d}$$

where $k_H + r_H = t_H$, $k_J + r_J = t_J$, Σ_H, Δ_H, Σ_J and Δ_J are diagonal, nonsingular and satisfy $\Sigma_H^2 + \Delta_H^2 = I$ and $\Sigma_J^2 + \Delta_J^2 = I$.

The following theorem then is the basis of the variant γ-iteration introduced in [6].

Theorem 2.4 *[6] For all $\gamma > \gamma_{mo}$, the matrix $\mathcal{Y}(\gamma)$ is positive semidefinite and* $\operatorname{rank} \mathcal{Y}(\gamma) = k_H + k_J$ *is constant. For all $\hat{\gamma} < \gamma < \gamma_{mo}$, either $\mathcal{Y}(\gamma)$ is not defined, or* $\operatorname{rank} \mathcal{Y}(\gamma) < k_H + k_J$, *or $\mathcal{Y}(\gamma)$ is not positive semidefinite.*

The problem of finding γ_{mo} thus reduces to the problem of finding the largest value of $\gamma (\geq \hat{\gamma})$ at which $\mathcal{Y}(\gamma)$ changes rank or fails to exist. This can be done, see [6], by applying a one-variable root-finding procedure to the eigenvalues of $\mathcal{Y}(\gamma)$. In this way a good approximation of γ_{mo} can be determined without explicitely forming the Hamiltonian matrices $H(\gamma), J(\gamma)$, while still using structure preserving methods [5]. Once γ_{mo} has been determined, it remains to construct a corresponding controller.

In order to avoid the hazards of solving ill-conditioned systems of equations, we will transform the formulas for the optimal controllers (12). For this, we will make frequent use of the following refactorization, that can be computed without forming explicit inverses or matrix products, [1–4].

Proposition 2.5 *Given a matrix product MP^{-1} with $M \in \mathbb{R}^{\eta\times\mu}$ and $P \in \mathbb{R}^{\mu\times\mu}$, there exist $\mathcal{P} \in \mathbb{R}^{\eta\times\eta}$ and $\mathcal{M} \in \mathbb{R}^{\eta\times\mu}$ such that*

$$MP^{-1} = \mathcal{P}^{-1}\mathcal{M}, \tag{15}$$

where the rows of $[\mathcal{P}, \mathcal{M}]$ span the η-dimensional left null space of $\begin{bmatrix} -M \\ P \end{bmatrix}$.

To construct a refactorization as in (15), observe that $MP^{-1} = \mathcal{P}^{-1}\mathcal{M}$ if and only if $\mathcal{P}M = \mathcal{M}P$ or, equivalently,

$$[\mathcal{P}, \mathcal{M}] \begin{bmatrix} -M \\ P \end{bmatrix} = 0.$$

So, each basis of the left nullspace gives rise to such a refactorization. A convenient way to calculate such a left null space and refactorization that was used in [1] (and in this paper) is to use a QR factorization of $\begin{bmatrix} -M \\ P \end{bmatrix}$, i.e.

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix} = \begin{bmatrix} -M \\ P \end{bmatrix}$$

with $R \in \mathbb{R}^{\mu\times\mu}$, $Q_{11} \in \mathbb{R}^{\eta\times\mu}$, $Q_{21} \in \mathbb{R}^{\mu\times\mu}$, $Q_{12} \in \mathbb{R}^{\eta\times\eta}$, and $Q_{22} \in \mathbb{R}^{\mu\times\eta}$. A suitable left null space is the row space of $[Q_{12}^T, Q_{22}^T]$ — one may use $\mathcal{P} = Q_{12}^T$, $\mathcal{M} = Q_{22}^T$.

3 Formulas for the (Sub)optimal Controller

This section discusses the construction of (sub)optimal H_∞-controllers as in (2) from γ_{mo} or an approximation $\gamma > \gamma_{mo}$ to it. The new approach is to reorganize the controller formulas (12) that use the computed data in (11a)–(11c) into a descriptor system form by avoiding numerical hazards where possible.

Using (4) and (5), we introduce

$$W_H = \begin{bmatrix} U_{21} & 0 \\ 0 & V_{12}\Sigma_{12}^{-1} \end{bmatrix}, \; W_J = \begin{bmatrix} U_{12} & 0 \\ 0 & V_{21}\Sigma_{21}^{-1} \end{bmatrix}, \quad (16)$$

and

$$\begin{aligned} \bar{B}_1 &= B_1 U_{21}, \quad \bar{B}_2 = B_2 V_{12}\Sigma_{12}^{-1}, \\ \bar{C}_1 &= U_{12}^T C_1, \quad \bar{C}_2 = \Sigma_{21}^{-1} V_{21}^T C_2. \end{aligned} \quad (17)$$

Note that in these formulas the inverses of the diagonal matrices Σ_{12} and Σ_{21} occur. These inverses can be formed in a numerically stable way. Ill-conditioning of these matrices, however, indicates that the resulting robust controller may be itself sensitive to small perturbations.

Using (5), we define $\bar{R}_H$ and $\bar{R}_J$ by

$$\bar{R}_H = W_H^T R_H W_H = \begin{bmatrix} \bar{D}_{11}^T \\ \bar{D}_{12}^T \end{bmatrix} \begin{bmatrix} \bar{D}_{11} & \bar{D}_{12} \end{bmatrix} - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\bar{R}_J = W_J^T R_J W_J = \begin{bmatrix} \bar{D}_{11} \\ \bar{D}_{21} \end{bmatrix} \begin{bmatrix} \bar{D}_{11}^T & \bar{D}_{21}^T \end{bmatrix} - \begin{bmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix},$$

and let $\begin{bmatrix} X_{H,1} \\ X_{H,2} \end{bmatrix}$, $\begin{bmatrix} X_{J,1} \\ X_{J,2} \end{bmatrix}$ be the orthonormal matrices defined in (13). Using (14a)–(14d) in Lemma 2.3, the unique symmetric positive semidefinite (stabilizing) ARE solutions X_H, X_J can be expressed as ([22, Theorem 17.1])

$$\begin{aligned} X_H &= X_{H,2} X_{H,1}^{-1} = U_H S_H C_H^{-1} U_H^T = U_H C_H^{-1} S_H U_H^T, \\ X_J &= X_{J,2} X_{J,1}^{-1} = U_J S_J C_J^{-1} U_J^T = U_J C_J^{-1} S_J U_J^T. \end{aligned}$$

Here, again, ill-conditioning of the diagonal matrices C_H, C_J may indicate high sensitivity of the controller. See [6] for more details.

With the quantities defined in (5) and (17) we introduce

$$\begin{aligned} F_M &= -\begin{bmatrix} \bar{D}_{11}^T \\ \bar{D}_{12}^T \end{bmatrix} \bar{C}_1 U_H C_H + \begin{bmatrix} \bar{B}_1^T \\ \bar{B}_2^T \end{bmatrix} U_H S_H, \\ L_M &= -C_J U_J^T \bar{B}_1 \begin{bmatrix} \bar{D}_{11}^T & \bar{D}_{21}^T \end{bmatrix} + S_J U_J^T \begin{bmatrix} \bar{C}_1^T & \bar{C}_2^T \end{bmatrix}. \end{aligned}$$

The matrices F, L, and Z in (11a)–(11c) can then be expressed as

$$F = W_H \hat{F}, \text{ where } \hat{F} = \bar{R}_H^{-1} F_M C_H^{-1} U_H^T =: \begin{matrix} \\ m_1 \\ m_2 \end{matrix} \overset{n}{\begin{bmatrix} \hat{F}_1 \\ \hat{F}_2 \end{bmatrix}}, \quad (18a)$$

$$L = \hat{L} W_J^T, \text{ where } \hat{L} = U_J C_J^{-1} L_M \bar{R}_J^{-1} =: \begin{bmatrix} \overset{p_1}{\hat{L}_1} & \overset{p_2}{\hat{L}_2} \end{bmatrix}, \quad (18b)$$

$$\begin{aligned} Z &= U_H C_H \hat{Z}^{-1} C_J U_J^T, \text{ where} \\ \hat{Z} &= C_J U_J^T U_H C_H - \gamma^{-2} S_J U_J^T U_H S_H. \end{aligned} \quad (18c)$$

Then the coefficient matrices $\hat{A}$,$\hat{B}_1$,$\hat{B}_2$,$\hat{C}_1$,$\hat{C}_2$ in the controller (2) can be expressed as

$$\begin{aligned} \hat{B}_2 &= Z(\bar{B}_2 + \hat{L}_1 \bar{D}_{12})\Sigma_{12} V_{12}^T \hat{D}_{12}, \\ \hat{B}_1 &= Z[(\bar{B}_2 + \hat{L}_1 \bar{D}_{12})(\Sigma_{12} V_{12}^T \hat{D}_{11} V_{21} \Sigma_{21}) - \hat{L}_2]\Sigma_{21}^{-1} V_{21}^T, \\ \hat{C}_2 &= -\hat{D}_{21} V_{21} \Sigma_{21} (\bar{C}_2 + \bar{D}_{21} \hat{F}_1), \\ \hat{C}_1 &= V_{12} \Sigma_{12}^{-1} [\hat{F}_2 - (\Sigma_{12} V_{12}^T \hat{D}_{11} V_{21} \Sigma_{21})(\bar{C}_2 + \bar{D}_{21} \hat{F}_1)], \\ \hat{A} &= A + \begin{bmatrix} \bar{B}_1 & \bar{B}_2 \end{bmatrix} \hat{F} - \hat{B}_1 V_{21} \Sigma_{21} (\bar{C}_2 + \bar{D}_{21} \hat{F}_1). \end{aligned}$$

Note that in these formulas still some unwanted inverses arise which we like to avoid. To do this, using Proposition 2.5, we can refactor the products $\bar{R}_H^{-1} F_M$ and $L_M \bar{R}_J^{-1}$ as

$$\bar{R}_H^{-1} F_M = \mathcal{F}_M \bar{\mathcal{R}}_H^{-1}, \qquad L_M \bar{R}_J^{-1} = \bar{\mathcal{R}}_J^{-1} \mathcal{L}_M.$$

Using the same notation as in [22], we partition $\mathcal{L}_M$ and $\mathcal{F}_M$ as

$$\mathcal{L}_M = \begin{bmatrix} \overset{p_1 - m_2}{\mathcal{L}_{M,11,\infty}} & \overset{m_2}{\mathcal{L}_{M,12,\infty}} & \overset{p_2}{\mathcal{L}_{M,2,\infty}} \end{bmatrix},$$

$$\mathcal{F}_M = \begin{matrix} m_1 - p_2 \\ p_2 \\ m_2 \end{matrix} \begin{bmatrix} \mathcal{F}_{M,11,\infty} \\ \mathcal{F}_{M,12,\infty} \\ \mathcal{F}_{M,2,\infty} \end{bmatrix}.$$

Then, in (18a)–(18c) we obtain new factors

$$\begin{aligned} \hat{F}_1 &= \begin{bmatrix} \mathcal{F}_{M,11,\infty} \\ \mathcal{F}_{M,12,\infty} \end{bmatrix} \bar{\mathcal{R}}_H^{-1} C_H^{-1} U_H^T, \\ \hat{F}_2 &= \mathcal{F}_{M,2,\infty} \bar{\mathcal{R}}_H^{-1} C_H^{-1} U_H^T, \\ \hat{L}_1 &= U_J C_J^{-1} \bar{\mathcal{R}}_J^{-1} \begin{bmatrix} \mathcal{L}_{M,11,\infty} & \mathcal{L}_{M,12,\infty} \end{bmatrix}, \\ \hat{L}_2 &= U_J C_J^{-1} \bar{\mathcal{R}}_J^{-1} \mathcal{L}_{M,2,\infty}. \end{aligned}$$

Let

$$D_1 = \tilde{U}_1 \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \tilde{V}_1^T$$

be the singular value decomposition of D_1 in (6). Define $\bar{D}_1$, $\bar{D}_2$, $\bar{D}_3$, $\bar{D}_4$, Δ_{13}, Δ_{23}, Δ_{31} and Δ_{32} in terms of $\tilde{U}_1$, $\tilde{V}_1$, D_1, D_2, D_3 and D_4 in (6) by

$$\begin{bmatrix} \bar{D}_1 & \bar{D}_2 \\ \bar{D}_3 & \bar{D}_4 \end{bmatrix} = \begin{bmatrix} \tilde{U}_1^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix} \begin{bmatrix} \tilde{V}_1 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \Sigma_1 & 0 & \Delta_{13} \\ 0 & 0 & \Delta_{23} \\ \Delta_{31} & \Delta_{32} & D_4 \end{bmatrix}.$$

If $d_1 = \operatorname{rank} D_1$, then $\Sigma_1 \in \mathbb{R}^{d_1,d_1}$, $\Delta_{13} \in \mathbb{R}^{d_1,p_2}$, $\Delta_{23} \in \mathbb{R}^{p_1-m_2-d_1,p_2}$, $\Delta_{31} \in \mathbb{R}^{m_2,d_1}$ and $\Delta_{32} \in \mathbb{R}^{m_2,m_1-p_2-d_1}$. Using these factorizations, we can rewrite

$$\begin{aligned} \tilde{D}_{11} &:= -(D_3 D_1^T \left(\gamma^2 I - D_1 D_1^T\right)^{-1} D_2 + D_4) \\ &= -(\bar{D}_3 \bar{D}_1^T (\gamma^2 I - \bar{D}_1 \bar{D}_1^T)^{-1} \bar{D}_2 + D_4) \end{aligned}$$

and define $\tilde{D}_{12}$ and $\tilde{D}_{21}$ via the Cholesky factorizations

$$\begin{aligned} \tilde{D}_{12}\tilde{D}_{12}^T &= I_{m_2} - D_3(\gamma^2 I - D_1^T D_1)^{-1} D_3^T \\ &= I_{m_2} - \bar{D}_3(\gamma^2 I - \bar{D}_1^T \bar{D}_1)^{-1} \bar{D}_3^T, \\ \tilde{D}_{21}^T\tilde{D}_{21} &= I_{p_2} - D_2^T(\gamma^2 I - D_1 D_1^T)^{-1} D_2 \\ &= I_{p_2} - \bar{D}_2^T(\gamma^2 I - \bar{D}_1 \bar{D}_1^T)^{-1} \bar{D}_2. \end{aligned}$$

By these factorizations and using M^+ to denote the Moore-Penrose pseudoinverse of M, we obtain

$$\begin{aligned} &(a)\ \tilde{D}_{11} = -\Delta_{31}\Sigma_1(\gamma^2 I - \Sigma_1^2)^+ \Delta_{13} + D_4, \\ &(b)\ \tilde{D}_{12}\tilde{D}_{12}^T = I_{m_2} - \Delta_{31}(\gamma^2 I - \Sigma_1^2)^+ \Delta_{31}^T - \gamma^{-2}\Delta_{32}\Delta_{32}^T, \\ &(c)\ \tilde{D}_{21}^T\tilde{D}_{21} = I_{p_2} - \Delta_{13}^T(\gamma^2 I - \Sigma_1^2)^+ \Delta_{13} - \gamma^{-2}\Delta_{23}^T\Delta_{23}. \end{aligned} \tag{19}$$

Note that it is not necessary to use Cholesky factorizations, mathematically, any factorizations satisfying (19) (b)–(c) can be used for $\tilde{D}_{12}$ and $\tilde{D}_{21}$.

In (19) we have replaced the inverses on the diagonal matrix $\gamma^2 I - \Sigma_1^2$ by Moore-Penrose generalized inverses. This allows to use the formulas even when for some γ value the matrix becomes singular.

By using (19) and the forms $\bar{D}_{12} = \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}$ and $\bar{D}_{21} = [0,\ I_{p_2}]$, we then have the reformulation of (11c) as

$$Z = U_H C_H \hat{Z}^{-1} C_J U_J^T,$$

where $\hat{Z} = C_J U_J^T U_H C_H - \gamma^{-2} S_J U_J^T U_H S_H$, and (12) becomes

$$\begin{aligned} (a)\quad \hat{D}_{11} &= V_{12}\Sigma_{12}^{-1}\tilde{D}_{11}\Sigma_{21}^{-1}V_{21}^T, \text{where} \\ \tilde{D}_{11} &= \Delta_{31}\Sigma_1(\gamma^2 I - \Sigma_1^2)^+\Delta_{13} + D_4, \\ (b)\quad \hat{D}_{12} &= V_{12}\Sigma_{12}^{-1}\tilde{D}_{12}, \text{where} \\ \tilde{D}_{12}\tilde{D}_{12}^T &= I_{m_2} - \Delta_{31}(\gamma^2 I - \Sigma_1^2)^+\Delta_{31}^T - \gamma^{-2}\Delta_{32}\Delta_{32}^T, \\ (c)\quad \hat{D}_{21} &= \tilde{D}_{21}\Sigma_{21}^{-1}V_{21}^T, \text{where} \\ \tilde{D}_{21}^T\tilde{D}_{21} &= I_{p_2} - \Delta_{13}^T(\gamma^2 I - \Sigma_1^2)^+\Delta_{13} - \gamma^{-2}\Delta_{23}^T\Delta_{23}, \\ (d)\ \hat{B}_2 &= Z(\bar{B}_2 + \hat{L}_1\bar{D}_{12})\tilde{D}_{12} = U_H C_H \hat{Z}^{-1}\bar{\mathcal{R}}_J^{-1}\cdot \\ &\quad (\bar{\mathcal{R}}_J C_J U_J^T \bar{B}_2 + \mathcal{L}_{M,12,\infty})\tilde{D}_{12}, \\ (e)\ \hat{B}_1 &= Z[(\bar{B}_2 + \hat{L}_1\bar{D}_{12})\tilde{D}_{11} - \hat{L}_2]\Sigma_{21}^{-1}V_{21}^T \\ &= U_H C_H \hat{Z}^{-1}\bar{\mathcal{R}}_J^{-1}[(\bar{\mathcal{R}}_J C_J U_J^T \bar{B}_2 \\ &\quad + \mathcal{L}_{M,12,\infty})\tilde{D}_{11} - \mathcal{L}_{M,2,\infty}]\Sigma_{21}^{-1}V_{21}^T, \\ (f)\ \hat{C}_2 &= -\tilde{D}_{21}(\bar{C}_2 + \bar{D}_{21}\hat{F}_1) \\ &= -\tilde{D}_{21}(\bar{C}_2 U_H C_H \bar{\mathcal{R}}_H + \mathcal{F}_{M,12,\infty})\bar{\mathcal{R}}_H^{-1}C_H^{-1}U_H^T, \\ (g)\ \hat{C}_1 &= V_{12}\Sigma_{12}^{-1}[\hat{F}_2 - \tilde{D}_{11}(\bar{C}_2 + \bar{D}_{21}\hat{F}_1)] = V_{12}\Sigma_{12}^{-1}\cdot \\ &\quad [\mathcal{F}_{M,2,\infty} - \tilde{D}_{11}(\bar{C}_2 U_H C_H \bar{\mathcal{R}}_H + \mathcal{F}_{M,12,\infty})]\cdot \\ &\quad \bar{\mathcal{R}}_H^{-1}C_H^{-1}U_H^T, \\ (h)\quad \hat{A} &= A + \begin{bmatrix} \bar{B}_1 & \bar{B}_2 \end{bmatrix}\hat{F} - (\hat{B}_1 V_{21}\Sigma_{21})(\bar{C}_2 + \bar{D}_{21}\hat{F}_1) \\ &= \Big\{ A U_H C_H \bar{\mathcal{R}}_H + \begin{bmatrix} \bar{B}_1 & \bar{B}_2 \end{bmatrix}\mathcal{F}_M - (\hat{B}_1 V_{21}\Sigma_{21})\cdot \\ &\quad (\bar{C}_2 U_H C_H \bar{\mathcal{R}}_H + \mathcal{F}_{M,12,\infty})\Big\}\bar{\mathcal{R}}_H^{-1}C_H^{-1}U_H^T. \end{aligned} \tag{20}$$

Theorem 17.1 in [22] states that the optimal controllers of the H_∞ control problem are obtained by composing

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}_1 y + \hat{B}_2\hat{u}, \\ u &= \hat{C}_1\hat{x} + \hat{D}_{11} y + \hat{D}_{12}\hat{u}, \\ \hat{y} &= \hat{C}_2\hat{x} + \hat{D}_{21} y, \end{aligned} \tag{21}$$

with a system

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}\hat{y}, \\ \hat{u} &= \tilde{C}\tilde{x} + \tilde{D}\hat{y}, \end{aligned}$$

whose transfer function $Q(s)$ has H_∞ norm less than γ. In any application, some such controllers are likely to be more robust, or less expensive, or more elegant, or otherwise more desirable than others. A convenient choice is $Q(s) = 0$, the "central controller" discussed in [22, Ch. 16–17] and [14].

This suggests the following procedure to avoid the remaining explicit matrix inverses. Refactor $C_H\hat{Z}^{-1}\bar{\mathcal{R}}_J^{-1}$ as

$$C_H(\bar{\mathcal{R}}_J\hat{Z})^{-1} = \mathcal{E}^{-1}\mathcal{C}_H,$$

as described in Proposition 2.5. Set $\check{x} = \bar{\mathcal{R}}_H^{-1} C_H^{-1} U_H^T \hat{x}$ and multiply the first equation in (21) by $\mathcal{C}_H \bar{\mathcal{R}}_J \hat{Z} C_H^{-1} U_H^T$ to obtain the descriptor system

$$\begin{aligned}\check{E}\dot{\check{x}} &= \check{A}\check{x} + \check{B}_1 y + \check{B}_2 \hat{u},\\ u &= \check{C}_1 \check{x} + \hat{D}_{11} y + \hat{D}_{12}\hat{u},\\ \hat{y} &= \check{C}_2 \check{x} + \hat{D}_{21} y,\end{aligned} \tag{22}$$

where $\hat{D}_{11}$, $\hat{D}_{12}$, and $\hat{D}_{21}$ are as in (20), and

$$\begin{aligned}
&(a)\quad \check{E} = \mathcal{E} C_H \bar{\mathcal{R}}_H,\\
&(b)\quad \check{B}_1 = \mathcal{C}_H[(\bar{\mathcal{R}}_J C_J U_J^T \bar{B}_2 + \mathcal{L}_{M,12,\infty})\tilde{D}_{11} - \mathcal{L}_{M,2,\infty}]\cdot\\
&\qquad\quad \Sigma_{21}^{-1} V_{21}^T,\\
&(c)\quad \check{B}_2 = \mathcal{C}_H \left(\bar{\mathcal{R}}_J C_J U_J^T \bar{B}_2 + \mathcal{L}_{M,12,\infty}\right)\tilde{D}_{12},\\
&(d)\quad \check{C}_1 = V_{12}\Sigma_{12}^{-1}\cdot\\
&\qquad\quad [\mathcal{F}_{M,2,\infty} - \tilde{D}_{11}(\bar{C}_2 U_H C_H \bar{\mathcal{R}}_H + \mathcal{F}_{M,12,\infty})],\\
&(e)\quad \check{C}_2 = -\tilde{D}_{21}(\bar{C}_2 U_H C_H \bar{\mathcal{R}}_H + \mathcal{F}_{M,12,\infty}),\\
&(f)\quad \check{A} = \mathcal{E} U_H^T \left(A U_H C_H \bar{\mathcal{R}}_H + \begin{bmatrix} \bar{B}_1 & \bar{B}_2 \end{bmatrix} \mathcal{F}_M\right)\\
&\qquad\quad -(\check{B}_1 V_{21}\Sigma_{21})(\bar{C}_2 U_H C_H \bar{\mathcal{R}}_H + \mathcal{F}_{M,12,\infty}).
\end{aligned} \tag{23}$$

A descriptor system expression for the central controller as suggested in [22, Ch. 16–17] and [14] is then of the form

$$\begin{aligned}\check{E}\dot{\check{x}} &= \check{A}\check{x} + \check{B}_1 y\\ u &= \check{C}_1 \check{x} + \check{D}_{11} y,\end{aligned} \tag{24}$$

where $\check{D}_{11} = \hat{D}_{11}$.

In this section we have presented new formulas for optimal H_∞ conrollers that avoid unnecessary numerical hazards. In the next section we illustrate the quality of the new formulas via several numerical examples.

4 Numerical examples

This section demonstrates the robustness of (22) in forming the suboptimal control for γ close to γ_{mo}. Note that in the presented examples for the computed values of γ_{mo} typically the solutions to the associated AREs do not exist, and as a consequence, the classical controller formulas cannot be applied. This is also the reason, why we do not present comparisons with other approaches. Unfortunately, we were also not successful in comparing our results for the examples presented here with the formulas presented in [15], since our implementation of these formulas did not produce correct results.

Example 4.1 This is Example 7.1 from [6] which first appeared in [22, p. 461]:

$$A = \begin{bmatrix} -a & 0 & 1 & -2 & 1\\ 0 & -100 & 0 & 0 & 0\\ 0 & 0 & 0 & -2a & a\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 3 & 2 \end{bmatrix},\quad B_1 = \begin{bmatrix}1\\0\\a\\0\\0\end{bmatrix},\quad B_2 = \begin{bmatrix}0\\-90\\0\\0\\1\end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\end{bmatrix},\quad D_{11} = \begin{bmatrix}0\\0\end{bmatrix},\quad D_{12} = \begin{bmatrix}0\\1\end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 0 & 1 & -2 & 1\end{bmatrix},\quad D_{21} = \begin{bmatrix}1\end{bmatrix},\quad D_{22} = \begin{bmatrix}0\end{bmatrix}.$$

In this example, γ_{mo} is independent of the choice of a. Using $a = 1$ and the variant γ-iteration described in [6], our experimental program determined $\gamma_{mo} = 7.853923684022$. With $\gamma = 7.8541$, our MATLAB implementation returns the central suboptimal controller (24) as

$$\check{E} = \begin{bmatrix} 0.1843 & -0.1737 & -0.0414 & -0.1018 & -0.0341\\ -0.1460 & 0.1434 & 0.4586 & -0.0903 & -0.0900\\ 0.1083 & -0.0368 & 0.0465 & 0.0126 & 0.0453\\ 0.2221 & 0.4605 & -0.1595 & 0.0046 & 0.0315\\ 0.2766 & -0.0515 & 0.1694 & 0.1360 & 0.2079 \end{bmatrix},$$

$$\check{A} = \begin{bmatrix} -3.7597 & 9.6660 & 0.0414 & -3.6630 & 8.3078\\ 5.7528 & -14.9391 & -0.4586 & 5.8541 & -12.5123\\ -2.3836 & 5.3517 & -0.0465 & -1.9928 & 4.5766\\ 9.7209 & -26.6163 & 0.1595 & 10.2078 & -22.4336\\ -6.4265 & 16.2923 & -0.1694 & -6.5070 & 13.7647 \end{bmatrix},$$

$$\check{B}_1 = \begin{bmatrix} -0.2857 & 0.2052 & -0.1335 & 0.0756 & 0.5193 \end{bmatrix}^T,$$

$$\check{C}_1 = \begin{bmatrix} 0.0937 & 0.2055 & -0.0000 & 0.1356 & -0.7939 \end{bmatrix},$$

$$\check{D}_{11} = [0].$$

The MATLAB robust control toolbox function `normhinf` [9] reports that the closed loop H_∞ norm is 7.8540366769208774.

Example 4.2 This is Example 4.1 and Example 7.2 from [6]. Consider the system

$$\left[\begin{array}{c|c|c} A & B_1 & B_2\\ \hline C_1 & D_{11} & D_{12}\\ \hline C_2 & D_{21} & 0 \end{array}\right] = \left[\begin{array}{cc|cc|c} -1 & 0 & 0 & 0 & 1\\ 0 & -1 & 0 & 0 & 1\\ \hline 1 & 0 & \frac{1}{2} & 0 & 0\\ 0 & 1 & 0 & \frac{1}{2} & 1\\ \hline 1 & 1 & 0 & 1 & 0 \end{array}\right].$$

The variant γ-iteration described in [6] determined $\gamma_{mo} = \hat{\gamma} = .5000000000000$ which agrees with the theoretical value to thirteen significant digits. With $\gamma = 0.50001$, our implementation returns the central suboptimal controller (24) as

$$\check{E} = \begin{bmatrix} -0.448419067323971 & 0.986059138740681 \\ -0.047683732968496 & 0.379074901512008 \end{bmatrix} \times 10^{-4},$$
$$\check{A} = \begin{bmatrix} -0.097230000237421 & 0.087478934743184 \\ -0.180766934371941 & 0.162712715362113 \end{bmatrix},$$
$$\check{B}_1 = \begin{bmatrix} -0.193452124650871 \\ -0.359503744256900 \end{bmatrix},$$
$$\check{C}_1 = \begin{bmatrix} -0.2514183866414180.226354558822012 \end{bmatrix},$$
$$\check{D}_{11} = [-0.5].$$

The function `normhinf` [9] reports that closed loop H_∞ norm is 0.500009995.

Example 4.3 This is Example 7.3 in [6]. Consider the system in Example 4.2 with the $(2, 2)$ element of B_1 set to one, i.e.,

$$\left[\begin{array}{c|c|c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{21} & 0 \end{array}\right] = \left[\begin{array}{cc|cc|c} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 1 \\ \hline 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 1 \\ \hline 1 & 1 & 0 & 1 & 0 \end{array}\right].$$

The variant γ-iteration described in [6] reports $\gamma_{mo} = 0.8062257748299$. With $\gamma = 0.80623$, our implementation returns the central suboptimal controller (24) as

$$\check{E} = \begin{bmatrix} -0.116338530296271 & -0.308766348091874 \\ 0.187484278723965 & -0.137335303455170 \end{bmatrix},$$
$$\check{A} = \begin{bmatrix} 0.188596106770156 & 0.745011728081016 \\ -0.103219543410531 & 0.479206276745429 \end{bmatrix},$$
$$\check{B}_1 = \begin{bmatrix} 0.066440383622532 \\ -0.376057188147526 \end{bmatrix},$$
$$\check{C}_1 = \begin{bmatrix} -0.005120459018602 & -0.213142135493424 \end{bmatrix},$$
$$\check{D}_{11} = [-0.5].$$

The function `normhinf` [9] reports that closed loop H_∞ norm is 0.80622598

Example 4.4 This is Example 7.4 in [6]. Let

$$\left[\begin{array}{c|c|c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hline C_2 & D_{21} & 0 \end{array}\right] = \left[\begin{array}{cc|cc|c} 2 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 & -2 \\ \hline 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ \hline 4 & -2 & 0 & 1 & 0 \end{array}\right].$$

In this example, $\hat{\gamma} = \gamma_{mo} = 3$. With $\gamma = 3.0001$, our implementation returns the central suboptimal controller (24) as

$$\check{E} = \begin{bmatrix} 0.205571073196335 & 0.034158098463962 \\ -0.402143396055301 & -0.066598384223367 \end{bmatrix},$$
$$\check{A} = \begin{bmatrix} -0.616633677960107 & -0.102994497996504 \\ 1.210370083410015 & 0.174898262642715 \end{bmatrix},$$
$$\check{B}_1 = \begin{bmatrix} -0.235063370281313 \\ 0.459795474769660 \end{bmatrix},$$
$$\check{C}_1 = \begin{bmatrix} 1.752972656836075 & 0.265994049023116 \end{bmatrix},$$
$$\check{D}_{11} = [1].$$

The function `normhinf` [9] reports that the closed loop H_∞ norm is 3.00000006.

Our last example is a standard textbook example.

Example 4.5 We consider a four-disk control system taken from [22]. Due to space limitations, we refer to [22, Example 19.4] for the data matrices. In [22] the optimal H_∞ norm is given as $\gamma_{opt} = 1.1272$ and a controller is computed for $\gamma = 1.2$. The MATLAB function `hinfsyn` computes $\gamma = 1.1292$ and constructs the controller accordingly. Our experimental code computes $\gamma = 1.1267$ and is able to construct a controller for that value.

5 Conclusion

In this paper we have introduced new formulas for computing optimal H_∞ controllers, that avoid all explicit matrix inverses (except for some diagonal matrices, i.e., row or column scalings) and many potential numerical difficulties that classical formulas may face. The formulas are based on the formulation used for the variant version of the γ-iteration presented in [6]. Since this γ-iteration has recently been extended to descriptor systems [16] we expect that these formulas can also be extended in a similar way. We have demonstrated the numerical properties of the new formulas with several numerical examples.

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