Numerical Methods for Linear Quadratic and H_∞ Control Problems

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Abstract

We discuss the numerical solution of linear quadratic optimal control problems and H_∞ control problems. A standard approach for these problems leads to solving algebraic Riccati equations or to the computation of deflating subspaces of structured matrix pencils. New structure preserving methods for these problems have been developed recently. These are faster than the conventional used methods and give results of full possible accuracy. The new methods can also be used for Riccati equations with an associated Hamiltonian matrix that has eigenvalues on the imaginary axis.

Keywords eigenvalue problem, deflating subspace, algebraic Riccati equation, Hamiltonian matrix, skew-Hamiltonian/Hamiltonian pencil
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1 Introduction and preliminaries

The numerical solution of linear quadratic control problems and H_∞ control problems is of great importance in the design of controllers, in particular when robust controllers are desired, [19, 27, 38, 33, 40].

The continuous time linear quadratic control problem has the following form. Minimize

$$\mathcal{S}(x,u) = \int_{t_0}^{\infty} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \tag{1}$$

subject to the differential-algebraic system

$$E\dot{x} = Ax + Bu, \qquad x(t_0) = x^0. \tag{2}$$

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Application of the maximum principle [39, 33] leads to the two-point boundary value problem of Euler-Lagrange equations

$$\mathcal{E}_c \begin{bmatrix} \dot{x} \\ \dot{\mu} \\ \dot{u} \end{bmatrix} = \mathcal{A}_c \begin{bmatrix} x \\ \mu \\ u \end{bmatrix}, \qquad x(t_0) = x^0, \qquad \lim_{t\to\infty} E^T \mu(t) = 0, \tag{3}$$

with the matrix pencil

$$\alpha\mathcal{E}_c - \beta\mathcal{A}_c := \alpha \begin{bmatrix} E & 0 & 0 \\ 0 & -E^T & 0 \\ 0 & 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} A & 0 & B \\ Q & A^T & S \\ S^T & B^T & R \end{bmatrix}. \tag{4}$$

The optimal solution $x(t)$ is required to be stable. If both E and R are nonsingular, then with $\eta := -E^T\mu$, (3) reduces to the two-point boundary value problem

$$\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \mathcal{H} \begin{bmatrix} x \\ \eta \end{bmatrix}, \qquad x(t_0) = x^0, \qquad \lim_{t\to\infty} \eta(t) = 0 \tag{5}$$

with the *Hamiltonian matrix*

$$\mathcal{H} = \begin{bmatrix} F & G \\ H & -F^T \end{bmatrix} := \begin{bmatrix} E^{-1}(A - BR^{-1}S) & E^{-1}BR^{-1}B^TE^{-T} \\ Q - SR^{-1}S^T & -(E^{-1}(A - BR^{-1}S))^T \end{bmatrix}. \tag{6}$$

The solution of the boundary value problem can be obtained in many different ways. For example, let Y be the stabilizing solution of the associated algebraic Riccati equation

$$0 = H + YF + F^TY - YGY. \tag{7}$$

Multiplying (6) from the left by

$$\begin{bmatrix} I & 0 \\ Y & I \end{bmatrix}$$

and changing the variables to

$$\begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix}$$

one obtains the decoupled Hamiltonian system

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} F - GY & G \\ 0 & -F^T + YG \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix}, \qquad x(t_0) = x_0, \qquad \lim_{t\to\infty} \xi(t) = 0, \tag{8}$$

from which the solution $[\, x^T, \, \xi^T \,]^T$ may be obtained by one reverse time and one forward time integration.

If E is singular and R is nonsingular then the system (3) represents a differential-algebraic system. Using $u(t) = -R^{-1}(S^Tx(t) + B^T\mu(t))$, system (3) reduces to

$$\mathcal{S} \begin{bmatrix} \dot{x} \\ \dot{\mu} \end{bmatrix} = \mathcal{H} \begin{bmatrix} x \\ \mu \end{bmatrix}, \qquad x(t_0) = x^0, \qquad \lim_{t\to\infty} E^T\mu(t) = 0, \tag{9}$$

with the reduced pencil

$$\alpha\mathcal{S} - \beta\mathcal{H} := \alpha \begin{bmatrix} E & 0 \\ 0 & -E^T \end{bmatrix} - \beta \begin{bmatrix} A - BR^{-1}S & -BR^{-1}B^T \\ Q - SR^{-1}S^T & (A - BR^{-1}S)^T \end{bmatrix} \tag{10}$$

In this case it is possible to write down a generalized Riccati equation but the relationship with solutions of the optimal control problem is lost or hidden. See [5, 25, 26, 33] for details.

If R is singular and E is nonsingular then the situation becomes more complicated. Although the boundary value problem remains well defined, the Riccati equation does not. The analysis of this case has been recently studied in [36, 24, 23] and numerical methods have been introduced in [4, 35, 41].

The case in which both E and R are singular has not been analyzed in full generality yet.

Note that the reduction to the form (5) may be still very ill-conditioned even if E or R are invertible. Hence it may happen that the transformed coefficient matrices in (6) are so corrupted by rounding errors that the solution obtained from them is of limited value. The same may happen if the solution is computed via the Riccati equation (7).

Hamiltonian matrices and Riccati equations of a similar structure occur in the H_∞ control problem. See, e.g., the recent monographs [21, 43]. The extended Hamiltonian pencils typically take the form

$$\alpha\mathcal{E}_h - \beta\mathcal{A}_h := \alpha \begin{bmatrix} E & 0 & 0 \\ 0 & -E^T & 0 \\ 0 & 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} A & \gamma^{-2}B_1B_1^T & B_2 \\ C^TC & A^T & 0 \\ 0 & B_2^T & I \end{bmatrix}. \tag{11}$$

In particular if E is nonsingular, then the reduced order system has the form (5). The main difference though is that in the linear quadratic control problem the matrix G is positive semidefinite, while in the H_∞ case it may be indefinite.

The Euler-Lagrange and Riccati equations, their solvability and their numerical solution has been the subject of numerous publications in recent years. See, e.g., [33, 11, 27, 40].

It was observed in [41] that it suffices to study the deflating subspaces of the pencil $(\mathcal{E}_c, \mathcal{A}_c)$ in (4) to solve the control problems. Suppose $(\mathcal{E}_c, \mathcal{A}_c)$ has an n-dimensional deflating subspace associated with eigenvalues in the left half plane. Let this subspace be spanned by the columns of a matrix $\mathcal{U}$, partitioned analogous to the pencil as

$$\mathcal{U} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}.$$

Then, if U_1 is invertible, the optimal control is a linear feedback of the form $u(t) = U_3U_1^{-1}x(t)$ and the solution of the associated Riccati equation is $Y = U_2U_1^{-1}E^{-1}$. See [33] for details.

Unfortunately, if E is singular, then such an n-dimensional deflating subspace in general does not exist. Under certain restrictions [33] we can complete the subspace to an n-dimensional subspace by adding appropriate eigenvectors and principal vectors associated with the eigenvalue ∞.

A feature of the pencils associated with the two-point boundary value problems is that they have algebraic structures which reflect the model and lead to a certain symmetry in the spectrum. Roundoff errors can destroy this symmetry leading to physically meaningless results unless the numerical method also preserves the algebraic structure of the pencil. Preserving algebraic structure also leads to more efficient as well as more accurate numerical methods.

Definition 1 *Let* $J := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, *where* I_n *is the* $n \times n$ *identity matrix.*

a) A matrix $\mathcal{H} \in \mathbf{R}^{2n\times 2n}$ *is* Hamiltonian *if* $(\mathcal{H}J)^T = \mathcal{H}J$. *The Lie Algebra of Hamiltonian matrices in* $\mathbf{R}^{2n\times 2n}$ *is denoted by* $\mathcal{H}_{2n}$.

b) A matrix $\mathcal{H} \in \mathbf{R}^{2n\times 2n}$ *is* skew-Hamiltonian *if* $(\mathcal{H}J)^T = -\mathcal{H}J$. *The Jordan algebra of skew-Hamiltonian matrices in* $\mathbf{R}^{2n\times 2n}$ *is denoted by* $\mathcal{SH}_{2n}$.

c) A matrix $\mathcal{S} \in \mathbf{R}^{n\times n}$ *is* symplectic *if* $\mathcal{S}J\mathcal{S}^T = J$. *The Lie group of symplectic matrices in* $\mathbf{R}^{n\times n}$ *is denoted by* $\mathcal{S}_{2n}$.

d) A matrix $\mathcal{U}_d \in \mathbf{R}^{2n\times 2n}$ *is* orthogonal symplectic *if* $\mathcal{U}_d J \mathcal{U}_d^T = J$ *and* $\mathcal{U}_d\mathcal{U}_d^T = I_{2n}$. *The compact Lie group of orthogonal symplectic matrices in* $\mathbf{R}^{n\times n}$ *is denoted by* $\mathcal{US}_{2n}$.

The reduced Euler-Lagrange equations (5) involve a Hamiltonian matrix and (9) involves a pencil with one skew-Hamiltonian and one Hamiltonian matrix.

The pencil (4) does not have this structure but many of the properties of Hamiltonian matrices carry over. We will discuss this in the next section.

Let us close the introductory remarks with some historical background on the numerical solution of the eigenvalue problems for matrices and pencils involving the structures in Definition 1.

The eigenproblem for Hamiltonian matrices has been a topic of research, since the landmark papers of Laub [28] and Paige/Van Loan [37]. While the Schur method proposed in [28] ignores the Hamiltonian structure and uses the standard QR algorithm to obtain the desired deflating subspace, the results in [37] suggest how to use the Hamiltonian structure. In [41] it was then discussed how to effectively use a staircase algorithm to treat the extended matrix pencil (4). But despite these important results and many other contributions, see [14, 29, 33] and the references therein, a completely satisfactory method is still an open problem. Such a method would be a numerically backward stable method, that has a complexity of $\mathcal{O}(n^3)$ and at the same time preserves the Hamiltonian structure.

There are two main reasons why this problem resisted solution. First of all one would need a triangular-like form under orthogonal symplectic similarity transformations from which the desired deflating subspaces can be read off. Such a Hamiltonian Schur form was first suggested in [37] but it is clear that not every Hamiltonian matrix has such a condensed form. The exact characterization when such a form exists was first proposed in [30] and finally proved in [34]. We will give a brief overview of these results in Section 3. The second difficulty arises from the fact that even if a Hamiltonian Schur form exists, it is not clear how to construct a method with the desired features to compute it numerically. It has been shown in [1] that a modification of standard QR-like methods to solve this problem is (except for special cases [15, 16]), in general hopeless, due to the missing reduction to a Hessenberg-like form. For this reason other methods like the multishift-method of [2] or the structured method of [9] were developed that do not follow the direct line of a standard QR-like method. Although these methods still do not fulfill all the requirements to a full extend, they come quite close to the optimal methods. We will review the method of [9] and indicate how it can be extended to skew-Hamiltonian/Hamiltonian pencils.

The outline of the paper is as follows. In Section 2 we describe a way to embed the extended Hamiltonian pencils into pencils with a skew-Hamiltonian/Hamiltonian structure and how this embedding can be interpreted from a system theoretic point of view. In Section 3 we briefly review the results on the existence of Hamiltonian Schur forms and in Section 4 we present numerical methods for the computation of the eigenvalues of Hamiltonian matrices as well as skew-Hamiltonian/Hamiltonian pencils. In Section 5 we then show how we can determine the deflating subspaces that we are interested in via structure preserving methods. and we discuss how the presented results can be extended to the complex case in Section 6.

2 Embedding of extended pencils

In this section we will show how we to endow the extended Hamiltonian pencil (4) with the structure of the pencil (10) by embedding the Euler-Lagrange equations (3) into a larger system. Introducing $\tilde{B} \in \mathbf{R}^{m \times n}$, $\tilde{R} \in \mathbf{R}^{m \times m}$, and an additional control vector $v \in \mathbf{R}^m$, we consider the extended dynamical system

$$E\dot{x} = Ax + Bu + \tilde{B}v, \qquad x(t_0) = x^0. \tag{12}$$

That is, the new control vector is given by $\begin{bmatrix} u \\ v \end{bmatrix}$. We also introduce a new cost functional

$$\mathcal{S}_e(x,u) = \int_{t_0}^{\infty} \begin{bmatrix} x(t) \\ u(t) \\ v(t) \end{bmatrix}^T \begin{bmatrix} Q & S & 0 \\ S^T & R & 0 \\ 0 & 0 & \tilde{R} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \\ v(t) \end{bmatrix} dt. \tag{13}$$

For this embedded system the Euler-Lagrange equations written in the appropriate order take the form

$$\mathcal{E}_e \begin{bmatrix} \dot{x} \\ \dot{u} \\ \dot{\mu} \\ \dot{v} \end{bmatrix} = \mathcal{A}_e \begin{bmatrix} x \\ u \\ \mu \\ v \end{bmatrix}, \qquad x(t_0) = x^0, \qquad \lim_{t\to\infty} E^T\mu(t) = 0, \tag{14}$$

with the extended skew-Hamiltonian/ Hamiltonian pencil

$$\alpha\mathcal{E}_e - \beta\mathcal{A}_e := \alpha \left[\begin{array}{cc|cc} E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & E^T & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] - \beta \left[\begin{array}{cc|cc} A & B & 0 & \tilde{B} \\ 0 & 0 & \tilde{B}^T & \tilde{R} \\ \hline -Q & -S & -A^T & 0 \\ -S^T & -R & -B^T & 0 \end{array}\right]. \tag{15}$$

In this embedding there is a lot of freedom. In principle we can choose $\tilde{R}$ and $\tilde{B}$ arbitrarily, but it seems appropriate to follow certain general rules. First of all we should choose them in such a way that the resulting pencil is a regular pencil, so that the associated two-point boundary value problem of differential-algebraic equations (14) has a unique solution for all consistently chosen initial values x^0, see [26]. If the original extended Hamiltonian pencil (4) has this property then this is easily achieved. Note that this regularity property can always be guaranteed via an appropriate preprocessing of the system, see [17, 26, 33]. Secondly, we should ensure that the embedded pencil has a structured Schur-like form as it was introduced for skew-Hamiltonian/Hamiltonian pencils in [32, 31]. Furthermore the problem of computing the desired invariant subspace should not become more ill-conditioned than that for the pencil (4). For a detailed discussion of the choice of $\tilde{B}, \tilde{R}$ see [7].

If we obey these principles then the embedding just means that we have added some eigenvalues at infinity to the system and increased the associated deflating subspace.

There is a certain philosophy behind this embedding trick. First of all we can view this extension as the converse operation to the reduction of the extended problem (3) to the problem (9). Furthermore if we consider a behavior approach, i.e., if we do not distinguish state and input variables, then the same global structure occurs for singular E of the form $\begin{bmatrix} \tilde{E} & 0 \\ 0 & 0 \end{bmatrix}$, if in the reduced system (10) we partition x and μ according to the partitioning in E. In view of these observations, the embedding seems a natural approach from a system theoretic point view. It allows better use of structure, and it avoids the inversion of matrices, which may lead to large numerical errors.

3 Hamiltonian Schur forms

In this section we briefly review the results on the existence of structured Schur forms for Hamiltonian matrices and skew-Hamiltonian/Hamiltonian pencils.

To simplify notation we use *eigenvalue* for eigenvalues of matrices and also for pairs $(\alpha, \beta) \neq (0, 0)$ for which the determinant of a matrix pencil $\alpha E - \beta A$ vanishes. These pairs are not unique. If $\beta \neq 0$ then we identify (α, β) with $(\frac{\alpha}{\beta}, 1)$ or $\lambda = \frac{\alpha}{\beta}$. Pairs $(\alpha, 0)$ with $\alpha \neq 0$ are called *infinite eigenvalues*.

A *quasi-triangular* matrix A is triangular with 1×1 or 2×2 blocks on the diagonal. We call a real matrix *Hamiltonian quasi-triangular* if it is Hamiltonian and has the form

$$\begin{bmatrix} F & G \\ 0 & -F^T \end{bmatrix},$$

where F is quasi-triangular in real Schur form [20]. Similarly we call a real matrix *skew-Hamiltonian quasi-triangular* if it is skew-Hamiltonian and has the form

$$\begin{bmatrix} F & G \\ 0 & F^T \end{bmatrix},$$

where F is quasi-triangular. If a Hamiltonian (skew-Hamiltonian) matrix $\mathcal{H}$ can be transformed into Hamiltonian quasi-triangular from by a similarity transformation with a matrix $\mathcal{U} \in \mathcal{US}_{2n}$, then we say that $\mathcal{U}^T \mathcal{H} \mathcal{U}$ has *Hamiltonian Schur form* (*skew-Hamiltonian Schur form*).

For Hamiltonian matrices that have no purely imaginary eigenvalues the existence of a Hamiltonian Schur form was proved in [37]. The general result was suggested in [30] and a proof based on a structured Hamiltonian Jordan form was recently given in [34]. Since the general result is quite technical, we only give here parts of the result proved in [34].

Theorem 2 *[34]*

Let $\mathcal{H}$ be a real Hamiltonian matrix, let $i\alpha_1, \ldots, i\alpha_\nu$ be its pairwise distinct nonzero purely imaginary eigenvalues and let U_k, $k = 1, \ldots, \nu$, be the associated invariant subspaces. Then the following are equivalent.

i) There exists a real symplectic matrix $\mathcal{S}$ such that $\mathcal{S}^{-1} \mathcal{H} \mathcal{S}$ is real Hamiltonian quasi-triangular.

ii) There exists a real orthogonal symplectic matrix $\mathcal{U}$ such that $\mathcal{U}^T\mathcal{H}\mathcal{U}$ is real Hamiltonian quasi-triangular.

iii) $U_k^H J U_k$ is congruent to J for all $k = 1, \ldots, \nu$, where J is always of the appropriate dimension.

This theorem gives necessary and sufficient conditions for the existence of a real Hamiltonian Schur form under orthogonal symplectic similarity transformations. On the other hand, there are Hamiltonian matrices for which these conditions do not hold, but nevertheless there exists a *nonsymplectic* similarity transformation to Hamiltonian quasi-triangular form. A class of such matrices are the matrices J of a size that is divisible by 4. Orthogonal symplectic similarity transformations do not change these matrices, hence they have no Hamiltonian quasi-triangular form under symplectic similarity transformations. But such matrices are similar to a Hamiltonian quasi triangular form under nonsymplectic transformations. As an example consider $J \in \mathbf{R}^{4\times 4}$. Set $V = [e_1, e_3, e_2, e_4]$, then $V^H J V = \text{diag}(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix})$ is Hamiltonian triangular.

Necessary and sufficient conditions for the existence of a Hamiltonian Schur form under nonsymplectic similarity transformations are given in the following theorem.

Theorem 3 *[34] A real Hamiltonian matrix $\mathcal{H}$ is similar to a real Hamiltonian triangular form if and only if the algebraic multiplicities of all purely imaginary eigenvalues with positive imaginary parts are even.*

A similar theorem applies to skew-Hamiltonian/Hamiltonian pencils.

Theorem 4 *[32, 31] Let $\alpha\mathcal{S} - \beta\mathcal{H}$ be a regular skew-Hamiltonian/ Hamiltonian pencil, let $i\alpha_1, \ldots, i\alpha_\nu$ be its pairwise distinct nonzero purely imaginary eigenvalues with algebraic multiplicities $p_1, \ldots, p_\nu$ and let U_k, $k = 1, \ldots, \nu$, be the associated deflating subspaces. Furthermore let p_∞ be the algebraic multiplicity of the eigenvalue infinity and let U_∞ be the associated deflating subspace. Then the following are equivalent.*

i) There exists a nonsingular matrix $\mathcal{P}$, such that

$$J\mathcal{P}^T J(\alpha\mathcal{S} - \beta\mathcal{H})\mathcal{P} = \alpha \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{11}^T \end{bmatrix} - \beta \begin{bmatrix} H_{11} & H_{12} \\ 0 & -H_{11}^T \end{bmatrix} \tag{16}$$

where S_{11} is upper triangular and H_{11} is quasi upper triangular.

ii) There exists a real orthogonal matrix $\mathcal{U}$ such that $J\mathcal{U}^T J(\alpha\mathcal{S} - \beta\mathcal{H})\mathcal{U}$ has the form (16).

iii) $U_k^H J\mathcal{S} U_k$ is congruent to J of appropriate dimension for all $k = 1, \ldots, \nu$. Furthermore if $p_\infty \neq 0$ then $U_\infty^T JHU_\infty$ is congruent to iJ of appropriate dimension.

Note that a necessary condition for iii) to hold is that all purely imaginary eigenvalue of $\mathcal{H}$ have even algebraic multiplicities. Similar results also exist for complex matrices and pencils, see [34] and [32, 31]. The results also demonstrate that whenever a structured triangular form exists, then it also exists under orthogonal transformations. This fact gives hope that these forms and therefore also the eigenvalues and deflating subspaces can be computed with structure preserving numerically stable methods. We discuss such methods for the computation of eigenvalues of Hamiltonian matrices and skew-Hamiltonian/Hamiltonian pencils in the remaining sections.

4 Eigenvalue computation

We have seen that (possibly by an appropriate embedding) the solution of our robust control problems leads to the problem of computing eigenvalues and deflating subspaces for Hamiltonian matrices or skew-Hamiltonian/Hamiltonian pencils. It is well-known that if $\mathcal{H}$ is a Hamiltonian matrix, $\mathcal{H}^2$ is a skew-Hamiltonian matrix. It is easier to compute the eigenvalues of a real skew-Hamiltonian matrix than those of a Hamiltonian matrix [42]. This suggests computing the eigenvalues of $\mathcal{H}$ by taking square roots of the eigenvalues of $\mathcal{H}^2$. This method was proposed in [42]. Unfortunately, in a worst case scenario one might obtain only half of the possible accuracy in the computed eigenvalues [15, 42]. An example demonstrating this was given in [42]. A way out of this dilemma was recently presented in [10].

Theorem 5 *[10] Let $\mathcal{H} \in \mathcal{H}_{2n}$. Then there exist $Q_1, Q_2 \in \mathcal{US}_{2n}$, such that*

$$Q_1^T \mathcal{H} Q_2 = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix}, \tag{17}$$

with H_{11} upper triangular and H_{22}^T quasi upper triangular. Furthermore the eigenvalues of $\mathcal{H}$ are the square roots of the eigenvalues of $-H_{11}H_{22}^T$.

Note that the resulting matrix in (17) is neither Hamiltonian nor similar to $\mathcal{H}$, but a simple calculation shows that both $Q_1^T \mathcal{H}^2 Q_1$ and $Q_2^T \mathcal{H}^2 Q_2$ are real skew-Hamiltonian quasi-triangular.

For skew-Hamiltonian/Hamiltonian pencils $\alpha\mathcal{S} - \beta\mathcal{H}$ of the form (10) or (15), we can construct similar methods. Roughly, the idea is to factor $\mathcal{S} = \mathcal{S}_1\mathcal{S}_2$ with $\mathcal{S}_1 = J\mathcal{S}_2^T J^T$ (e.g., for (10), $\mathcal{S}_1 = \begin{bmatrix} I & 0 \\ 0 & E^T \end{bmatrix}$) and to apply the previous procedure formally to the Hamiltonian matrix

$$\mathcal{S}_1^{-1}\mathcal{H}\mathcal{S}_2^{-1}$$

without ever forming the product or the inverses.

Theorem 6 *[7] If the skew-Hamiltonian/Hamiltonian pencil*

$$\alpha\mathcal{S}_1\mathcal{S}_2 - \beta\mathcal{H} := \alpha \begin{bmatrix} I & 0 \\ 0 & E^T \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix} - \beta \begin{bmatrix} F & G \\ H & -F^T \end{bmatrix} \tag{18}$$

is regular, then there exist orthogonal matrices Q_3, Q_4 and orthogonal symplectic matrices Q_1, Q_2, such that

$$\begin{aligned} Q_3^T \mathcal{S}_1 Q_1 &= \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix}, \\ Q_2^T \mathcal{S}_2 Q_4 &= \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}, \\ Q_3^T \mathcal{H} Q_4 &= \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix}, \end{aligned} \tag{19}$$

where $S_{11}, T_{11}, H_{11}, S_{22}^T, T_{22}^T$ are upper triangular and H_{22}^T is quasi upper triangular. Furthermore, the finite eigenvalues of $\alpha\mathcal{S} - \beta\mathcal{H}$ are

1. the square roots of the finite eigenvalues of $\alpha S_{11}S_{22}^T + \beta H_{11}T_{11}^{-1}T_{22}^{-T}H_{22}^T$*;*

2. or, equivalently, the eigenvalues of $\alpha \begin{bmatrix} T_{22}^T T_{11} & 0 \\ 0 & S_{11}S_{22}^T \end{bmatrix} - \beta \begin{bmatrix} 0 & -H_{22}^T \\ H_{11} & 0 \end{bmatrix}$.

The proof of this result as well as the proof of Theorem 5 amount to algorithms for the computation of (17) and (19). The reduction procedures are based on a finite elimination procedure that brings all of the diagonal blocks except H_{22} to triangular form. The block H_{22} reduces to lower Hessenberg form. This initial reduction is then followed by the periodic QR-algorithm or QZ-algorithm [12, 22] applied to $-H_{11}H_{22}^T$ or

$$-S_{11}^{-1}H_{11}T_{11}^{-1}T_{22}^{-T}H_{22}^T S_{22}^{-T}, \tag{20}$$

respectively.

The periodic QR-algorithm applied to $-H_{11}H_{22}^T$ yields real orthogonal transformation matrices $U, V \in \mathbf{R}^{n\times n}$ such that $U^T H_{11} V$ is upper triangular and $(U^T H_{22} V)^T$ is quasi upper triangular. Analogously the periodic QZ-algorithm applied to (20) yields real orthogonal transformation matrices $P, Q, U, V, Y, Z \in \mathbf{R}^{n\times n}$, such that $P^T S_{11} Q$, $P^T H_{11} U$, $V^T T_{11} U$, $W^T T_{22}^T V$, $Q^T S_{22}^T Y$ are upper triangular, and $W^T H_{22}^T Y$ is quasi upper triangular. The 2×2 blocks are associated only with nonsingular blocks in S_{11}, S_{22}, T_{11}, T_{22}.

After these forms have been computed, we can compute the eigenvalues of $\mathcal{H}$ or $\alpha\mathcal{S} - \beta\mathcal{H}$, respectively by solving 1×1 or 2×2 eigenvalue problems and taking square roots. For algorithmic details and a detailed error analysis see [7, 10].

To demonstrate the efficiency of this approach we present numerical examples for the Hamiltonian matrix case. The numerical tests were performed using IEEE double precision arithmetic with machine precision $\varepsilon \approx 2.2204 \times 10^{-16}$ on a HP Model 712/60 workstation with operating system HP-UX 9.0. We used the HP-UX Fortran 77 compiler invoked by `f77`. The programs were compiled using standard optimization.

We compared the following methods:

- **URVPSD**, the method based on Theorem 5 as suggested in [10],
- **SQRED**, Van Loan's square reduced method [42] as implemented in [6],
- **LAPACK**, the nonsymmetric eigenproblem solver DGEEVX from LAPACK [3].

All subroutines use the BLAS and LAPACK [3] compiled from Fortran source with `f77 -O`. The implementations of URVPSD and SQRED are not block-oriented.

Example 1 [42, Example 2] Let $F = \mathrm{diag}(1, 10^{-2}, 10^{-4}, 10^{-6}, 10^{-8})$, and let H be the Hamiltonian matrix obtained by

$$H = U^T \begin{bmatrix} F & 0 \\ 0 & -F^T \end{bmatrix} U,$$

with $U \in \mathcal{US}_{2n}$ randomly generated by five symplectic rotations and five reflectors. Thus,

$$\sigma(H) = \{\pm 1, \pm 10^{-2}, \pm 10^{-4}, \pm 10^{-6}, \pm 10^{-8}\}.$$

Table 1 shows the absolute errors in the eigenvalue approximations computed by the three methods.

λ	URVPSD	SQRED	LAPACK
1	0	0	7.8×10^{-16}
10^{-2}	5.5×10^{-16}	5.5×10^{-16}	5.0×10^{-17}
10^{-4}	1.6×10^{-18}	1.6×10^{-14}	2.6×10^{-18}
10^{-6}	1.0×10^{-18}	1.5×10^{-11}	8.4×10^{-18}
10^{-8}	3.1×10^{-17}	2.2×10^{-9}	4.7×10^{-17}

Table 1: Example 1, absolute errors $|\lambda - \tilde{\lambda}|$.

Table 1 demonstrates that SQRED calculates large magnitude eigenvalues to full precision (apart from the effects of eigenvalue ill-conditioning) but that small magnitude eigenvalues to only half precision. LAPACK and URVPSD [10] calculate all eigenvalues to full precision.

Example 2 We also tested the three methods for randomly generated Hamiltonian matrices with entries distributed uniformly in the interval $[-1, 1]$. Since the eigenvalue distribution for these examples usually behaves "nicely", the eigenvalues computed by either of the methods are computed to almost the same accuracy. We give the CPU times for $2n \times 2n$ examples for several sizes of n. For each size of n, we computed 100 examples. The values given in Table 2 are the mean values of the CPU times measured on a HP Model 712/60 work station.

n	URVPSD	SQRED	LAPACK
25	0.092	0.061	0.142
50	0.56	0.34	0.77
75	1.72	1.03	2.36
100	3.95	2.41	5.30
125	7.36	4.66	10.07
150	12.33	7.99	17.36
175	19.52	12.53	27.79
200	28.61	18.51	41.44

Table 2: Example 2, CPU times in seconds.

Table 2 shows that URVPSD and SQRED are much faster than the standard QR algorithm. The speed-ups are roughly proportional to the flop counts. There is a little overhead which causes both methods to be slightly slower than to be expected from the flop counts, though. This is due to the fact that these methods are more complex as far as index handling, memory access, and subroutine calls are concerned.

We have seen that it is possible to use the algebraic structure of Hamiltonian matrices and skew-Hamiltonian/Hamiltonian pencils effectively to speed up the computation of eigenvalues while still achieving full possible accuracy. Unfortunately this new approach is not perfect. We would like to have the Hamiltonian Schur form, since it would give us the eigenvalues and also the deflating subspaces. Currently the only other candidate for an optimal algorithm, the multishift algorithm of [2], sometimes has convergence problems.

For the computation of the deflating subspaces we now use another embedding procedure. These ideas are presented in the next section.

5 Invariant subspace computation for Hamiltonian matrices

In this section we discuss structure preserving methods to compute the invariant subspaces of Hamiltonian matrices. This approach can also be applied to general matrices, so we present it in general and then show how it specializes for Hamiltonian matrices. The description of the treatment of skew-Hamiltonian/Hamiltonian pencils is similar, but technically involved. For this reason we present here only the method for the Hamiltonian matrix case and refer the reader to the forthcoming paper [7] for the pencil case.

Let $\lambda_-(A), \lambda_+(A), \lambda_0(A)$ denote the spectra in the open left half plane, in the open right half plane and on the imaginary axis, of a matrix A, respectively. The associated invariant subspaces are denoted by $\mathrm{Inv}_-(A), \mathrm{Inv}_+(A)$, $\mathrm{Inv}_0(A)$, respectively.

Let $A \in \mathbf{R}^{n\times n}$. If

$$B = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}, \tag{21}$$

then

$$\begin{aligned} \lambda(B) &= \lambda(A) \cup \lambda(-A), \\ \lambda_0(B) &= \lambda_0(A) \cup \lambda_0(A), \\ \lambda_+(B) &= \lambda_+(A) \cup \lambda_+(-A) \;=\; \lambda_+(A) \cup (-\lambda_-(A)), \\ \lambda_-(B) &= \lambda_-(A) \cup \lambda_-(-A) \;=\; (-\lambda_+(A)) \cup \lambda_-(A) \;=\; -\lambda_+(B). \end{aligned} \tag{22}$$

Furthermore we obtain the following relations for the invariant subspaces of A and B.

Theorem 7 *[9] Let $A \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{2n\times 2n}$ be related as in (21) and let $\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \in \mathbf{R}^{2n\times n}$, $Q_1, Q_2 \in \mathbf{R}^{n\times n}$, have orthonormal columns, such that*

$$B \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} R, \tag{23}$$

where

$$\lambda_+(B) \subseteq \lambda(R) \subseteq \lambda_+(B) \cup \lambda_0(B). \tag{24}$$

Then

$$\mathrm{range}\{Q_1 + Q_2\} = \mathrm{Inv}_+(A) + \mathcal{N}_1, \quad \textit{where} \quad \mathcal{N}_1 \subseteq \mathrm{Inv}_0(A), \tag{25}$$

$$\mathrm{range}\{Q_1 - Q_2\} = \mathrm{Inv}_-(A) + \mathcal{N}_2, \quad \textit{where} \quad \mathcal{N}_2 \subseteq \mathrm{Inv}_0(A). \tag{26}$$

Moreover, if we partition R as

$$R = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}, \; \textit{where} \; \lambda(R_{11}) = \lambda_+(B), \tag{27}$$

and, accordingly, $Q_1 = \begin{bmatrix} Q_{11} & Q_{12} \end{bmatrix}$, $Q_2 = \begin{bmatrix} Q_{21} & Q_{22} \end{bmatrix}$, *then*

$$B \begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} = \begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} R_{11}, \tag{28}$$

and there exists an orthogonal matrix Z *such that*

$$\begin{aligned} \frac{\sqrt{2}}{2}(Q_{11} + Q_{21}) &= \begin{bmatrix} 0 & P_+ \end{bmatrix} Z, \\ \frac{\sqrt{2}}{2}(Q_{11} - Q_{21}) &= \begin{bmatrix} P_- & 0 \end{bmatrix} Z, \end{aligned} \tag{29}$$

where P_+, P_- *are orthogonal bases of* $\text{Inv}_+(A)$, $\text{Inv}_-(A)$, *respectively.*

In the case of a Hamiltonian matrix $\mathcal{H} = \begin{bmatrix} F & G \\ H & -F^T \end{bmatrix}$ again consider the block matrix

$$\mathcal{B} = \begin{bmatrix} 0 & \mathcal{H} \\ \mathcal{H} & 0 \end{bmatrix}. \tag{30}$$

If

$$\mathcal{P} = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix}, \tag{31}$$

then

$$\tilde{\mathcal{B}} := \mathcal{P}^T \mathcal{B} \mathcal{P} = \begin{bmatrix} 0 & F & 0 & G \\ F & 0 & G & 0 \\ 0 & H & 0 & -F^T \\ H & 0 & -F^T & 0 \end{bmatrix} \tag{32}$$

is again Hamiltonian.

Theorem 8 *[9] Let* $\mathcal{H} \in H_{2n}$ *and let* $\mathcal{B}$ *be as in (30). Then there exists* $\mathcal{U} \in U_{4n}$, *such that*

$$\mathcal{U}^T \mathcal{B} \mathcal{U} = \begin{bmatrix} R & D \\ 0 & -R^T \end{bmatrix} =: \mathcal{R} \tag{33}$$

is in Hamiltonian quasi-triangular form and $\lambda_-(R) = \emptyset$. *Moreover,* $\mathcal{U} = \mathcal{P}\mathcal{W}$ *with* $\mathcal{W} \in US_{4n}$, *and*

$$\mathcal{R} = \mathcal{W}^T \tilde{\mathcal{B}} \mathcal{W}, \tag{34}$$

i.e., $\mathcal{R}$ *is the Hamiltonian quasi-triangular form of the Hamiltonian matrix* $\tilde{\mathcal{B}}$. *Furthermore, if* $\mathcal{H}$ *has no purely imaginary eigenvalues, then* R *has only eigenvalues with positive real part.*

A constructive proof leading to the following algorithm for this result is given in [9], but the theorem also follows from Theorems 3 and 2.

Algorithm 1

Input: *A Hamiltonian matrix* $\mathcal{H} \in H_{2n}$ *having an* n*-dimensional Lagrangian invariant subspace.*

Output: $Y \in \mathbf{R}^{2n\times n}$, *with* $Y^TY = I_n$, *such that the columns of* Y *span a Lagrange invariant subspace.*

Step 1 *Apply Algorithm 2 of [10] to* $\mathcal{H}$ *and compute orthogonal symplectic matrices* $Q_1, Q_2 \in \mathcal{US}_{2n}$ *such that*

$$Q_1^T \mathcal{H} Q_2 = \begin{bmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{bmatrix}$$

is the decomposition (17).

Step 2 *Determine an orthogonal matrix* Q_3, *such that*

$$Q_3^T \begin{bmatrix} 0 & -H_{22}^T \\ H_{11} & 0 \end{bmatrix} Q_3 = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T_{22} & T_{23} \\ 0 & 0 & T_{33} \end{bmatrix}$$

is in real Schur form ordered such that the eigenvalues of T_{11} *have positive real part, the eigenvalues of* T_{22} *have zero real part, and the eigenvalues of* T_{33} *have negative real part.*

Step 3 *Use the orthogonal symplectic reordering scheme of [16] to determine an orthogonal symplectic matrix* $V \in \mathcal{US}_{4n}$ *such that with*

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} := \begin{bmatrix} Q_1Q_3 & 0 \\ 0 & Q_2Q_3 \end{bmatrix} V.$$

At this point we have the Hamiltonian quasi-triangular form

$$U^T \mathcal{B} U = \begin{bmatrix} F_{11} & F_{12} & G_{11} & G_{12} \\ 0 & F_{22} & G_{21} & G_{22} \\ 0 & 0 & -F_{11}^T & 0 \\ 0 & 0 & -F_{12}^T & -F_{22}^T \end{bmatrix},$$

where F_{11}, F_{22} *are quasi upper triangular with eigenvalues only in the closed right half plane.*

Step 4 *Set* $\hat{Y} := \frac{\sqrt{2}}{2}(U_{11} - U_{21})$. *Compute* Y, *an orthogonal basis of* $\mathrm{range}\{\hat{Y}\}$, *using any numerically stable orthogonalization scheme, for example a rank-revealing QR-decomposition; see, e.g., [18].*

End

The estimated computational cost for this algorithm is given in Table 3.

These numbers compare with $203n^3$ flops for the computation of the same invariant subspace via the standard QR-algorithm as suggested in [28].

Clearly we can obtain the desired solution of the Riccati equation (if it exists) from the invariant subspace but it is also possible to get it directly from $\hat{Y}$. See [9] for details.

A detailed description of this algorithm, an error and perturbation analysis as well as a comparison of different Riccati solvers are given in [9].

Step	1	2	3	4	total
flops	103 n^3	9 n^3	9 n^3	21 n^3	142 n^3

Table 3: Flop counts for Algorithm 1

6 Complex problems

Complex versions of Theorems 3 and 6 are given in [34], and a unitary-symplectic method for the Hamiltonian eigenproblem is presented in [13]. The algorithms described in the previous sections do not directly carry over to the case of complex Hamiltonian matrices. Another simple embedding, however, yields methods for the complex case [8].

Since for every complex Hamiltonian matrix $\mathcal{H}$, $i\mathcal{H}$ is complex skew-Hamiltonian, it suffices to study skew-Hamiltonian matrices. Let $N = N_1 + iN_2$ be a complex skew-Hamiltonian matrix with a real skew-Hamiltonian matrix $N_1 = \begin{bmatrix} F_1 & D_1 \\ G_1 & F_1^T \end{bmatrix}$ and a real Hamiltonian matrix $N_2 = \begin{bmatrix} F_2 & D_2 \\ G_2 & -F_2^T \end{bmatrix}$. Then with the unitary matrix

$$Q_{2n} := \frac{\sqrt{2}}{2} \begin{bmatrix} I_{2n} & iI_{2n} \\ I_{2n} & -iI_{2n} \end{bmatrix}, \tag{35}$$

and the permutation matrix $\mathcal{P}$ of (31) we obtain the real skew-Hamiltonian matrix

$$\begin{aligned} \mathcal{N} &:= \mathcal{P}^H Q_{2n}^H \operatorname{diag}(N, \overline{N}) Q_{2n} \mathcal{P} = \begin{bmatrix} F_1 & -F_2 & D_1 & -D_2 \\ F_2 & F_1 & D_2 & D_1 \\ G_1 & -G_2 & F_1^T & F_2^T \\ G_2 & G_1 & -F_2^T & F_1^T \end{bmatrix} \\ &=: \begin{bmatrix} \mathcal{F} & \mathcal{D} \\ \mathcal{G} & \mathcal{F}^T \end{bmatrix}. \end{aligned} \tag{36}$$

for which we can easily, see [42], obtain the real skew-Hamiltonian quasi-triangular form

$$W^T \mathcal{N} W = \begin{bmatrix} R & T \\ 0 & R^T \end{bmatrix} =: \mathcal{R}, \tag{37}$$

where $R \in \mathbf{R}^{2n\times 2n}$ is quasi upper triangular, $T = -T^T$, and $W \in \mathcal{US}_{4n}$ is real orthogonal symplectic. As in Section 5, we can then determine the desired subspaces. See [8] for details.

7 Conclusion

We have given a survey on recent results concerning the existence of Schur like forms for Hamiltonian matrices and skew-Hamiltonian/Hamiltonian pencils. Furthermore we have discussed structure preserving numerical methods for the computation of invariant and deflating subspaces for these matrices and pencils. The methods can also be used for problems with purely imaginary eigenvalues, although there are still some open problems to be settled. The presented ideas allow a universal treatment of linear quadratic and H_∞ problems and analogous results and methods are also available for discrete-time problems.

References

[1] G. S. Ammar and V. Mehrmann. On Hamiltonian and symplectic Hessenberg forms. *Linear Algebra Appl.*, 149:55–72, 1991.

[2] G.S. Ammar, P. Benner, and V. Mehrmann. A multishift algorithm for the numerical solution of algebraic Riccati equations. *Electr. Trans. Num. Anal.*, 1:33–48, 1993.

[3] E. Anderson, Z. Bai, C. Bischof, J. Demmel, J. Dongarra, J. Du Croz, A. Greenbaum, S. Hammarling, A. McKenney, S. Ostrouchov, and D. Sorensen. *LAPACK Users' Guide*. SIAM, Philadelphia, PA, second edition, 1995.

[4] W.F. Arnold, III and A.J. Laub. Generalized eigenproblem algorithms and software for algebraic Riccati equations. *Proc. IEEE*, 72:1746–1754, 1984.

[5] D.J. Bender and A.J. Laub. The linear-quadratic optimal regulator for descriptor systems. *IEEE Trans. Automat. Control*, AC-32:672–688, 1987.

[6] P. Benner, R. Byers, and E. Barth. HAMEV and SQRED: Fortran 77 subroutines for computing the eigenvalues of Hamiltonian matrices using Van Loan's square reduced method. Technical Report SFB393/96-06, Fak. f. Mathematik, TU Chemnitz–Zwickau, 09107 Chemnitz, FRG, 1996. Available from `http://www.tu-chemnitz.de/sfb393/sfb96pr.html`.

[7] P. Benner, R. Byers, V. Mehrmann, and H. Xu. Numerical computation of deflating subspaces of embedded Hamiltonian and symplectic pencils. In preparation.

[8] P. Benner, V. Mehrmann, and H. Xu. A note on the numerical solution of the complex Hamiltonian and skew-Hamiltonian eigenvalue problems. In preparation.

[9] P. Benner, V. Mehrmann, and H. Xu. A new method for computing the stable invariant subspace of a real Hamiltonian matrix. *J. Comput. Appl. Math.*, 86:17–43, 1997.

[10] P. Benner, V. Mehrmann, and H. Xu. A numerically stable, structure preserving method for computing the eigenvalues of real Hamiltonian or symplectic pencils. *Numer. Math.*, 78(3):329–358, 1998.

[11] S. Bittanti, A. Laub, , and J. C. Willems, editors. *The Riccati Equation*. Springer-Verlag, Berlin, 1991.

[12] A. Bojanczyk, G.H. Golub, and P. Van Dooren. The periodic Schur decomposition; algorithms and applications. In *Proc. SPIE Conference, vol. 1770*, pages 31–42, 1992.

[13] A. Bunse-Gerstner and H. Faßbender. A Jacobi-like method for solving algebraic Riccati equations on parallel computers. *IEEE Trans. Automat. Control*, 42(8), 1997.

[14] A. Bunse-Gerstner, R. Byers, and V. Mehrmann. Numerical methods for algebraic Riccati equations. In S. Bittanti, editor, *Proc. Workshop on the Riccati Equation in Control, Systems, and Signals*, pages 107–116, Como, Italy, 1989.

[15] R. Byers. *Hamiltonian and Symplectic Algorithms for the Algebraic Riccati Equation*. PhD thesis, Cornell University, Dept. Comp. Sci., Ithaca, NY, 1983.

[16] R. Byers. A Hamiltonian QR-algorithm. *SIAM J. Sci. Statist. Comput.*, 7:212–229, 1986.

[17] R. Byers, T. Geerts, and V. Mehrmann. Descriptor systems without controllability at infinity. *SIAM J. Cont.*, 35(2):462–479, 1997.

[18] T. Chan. Rank revealing QR factorizations. *Linear Algebra Appl.*, 88/89:67–82, 1987.

[19] L. Dai. *Singular Control Systems.* Number 118 in Lecture Notes in Control and Information Sciences. Springer-Verlag, Berlin, 1989.

[20] G.H. Golub and C.F. Van Loan. *Matrix Computations.* Johns Hopkins University Press, Baltimore, third edition, 1989.

[21] M. Green and D.J.N Limebeer. *Linear Robust Control.* Prentice-Hall, Englewood Cliffs, NJ, 1995.

[22] J.J. Hench and A.J. Laub. Numerical solution of the discrete-time periodic Riccati equation. *IEEE Trans. Automat. Control*, 39:1197–1210, 1994.

[23] V. Ionescu, C. Oară, and M. Weiss. *Generalized Riccati theory. A Popov function approach.* John Wiley & Sons, Chichester, 1998.

[24] V. Ionescu and M. Weiss. Continuous and discrete-time Riccati theory: A Popov function approach. *Linear Algebra Appl.*, 193:173–209, 1993.

[25] P. Kunkel and V. Mehrmann. Numerical solution of Riccati differential algebraic equations. *Linear Algebra Appl.*, 137/138:39–66, 1990.

[26] P. Kunkel and V. Mehrmann. The linear quadratic control problem for linear descriptor systems with variable coefficients. *Math. Control, Signals, Sys.*, 10:247–264, 1997.

[27] P. Lancaster and L. Rodman. *The Algebraic Riccati Equation.* Oxford University Press, Oxford, 1995.

[28] A.J. Laub. A Schur method for solving algebraic Riccati equations. *IEEE Trans. Automat. Control*, AC-24:913–921, 1979. (See also *Proc. 1978 CDC (Jan. 1979)*, pp. 60-65).

[29] A.J. Laub. Invariant subspace methods for the numerical solution of Riccati equations. In S. Bittanti, A.J. Laub, and J.C. Willems, editors, *The Riccati Equation*, pages 163–196. Springer-Verlag, Berlin, 1991.

[30] W.-W. Lin and T.-C. Ho. On Schur type decompositions for Hamiltonian and symplectic pencils. Technical report, Institute of Applied Mathematics, National Tsing Hua University, Taiwan, 1990.

[31] C. Mehl. *Compatible Lie and Jordan algebras and applications to structured matrices and pencils.* Dissertation, Fakultät für Mathematik, TU Chemnitz, 09107 Chemnitz (FRG), 1998.

[32] C. Mehl. Condensed forms for skew-Hamiltonian/Hamiltonian pencils. Technical Report SFB393/98-8, Fak. f. Mathematik, TU Chemnitz, 09107 Chemnitz, FRG, 1998. Available from `http://www.tu-chemnitz.de/sfb393/sfb98pr.html`.

[33] V. Mehrmann. *The Autonomous Linear Quadratic Control Problem, Theory and Numerical Solution.* Number 163 in Lecture Notes in Control and Information Sciences. Springer-Verlag, Heidelberg, July 1991.

[34] V. Mehrmann and H. Xu. Canonical forms for Hamiltonian and symplectic matrices and pencils. Technical Report SFB393/98-7, Fak. f. Mathematik, TU Chemnitz, 09107 Chemnitz, FRG, 1998. Available from `http://www.tu-chemnitz.de/sfb393/sfb98pr.html`.

[35] C. Oară. Proper deflating subspaces: Properties, algorithms and applications. *Numer. Algorithms*, 7:355–373, 1994.

[36] C. Oară. *Generalized Riccati Theory: A Ppopov Function Approach.* PhD thesis, Polytechnic University Bucharest, Bucharest, Romania, 1995.

[37] C.C. Paige and C.F. Van Loan. A Schur decomposition for Hamiltonian matrices. *Linear Algebra Appl.*, 14:11–32, 1981.

[38] P.H. Petkov, N.D. Christov, and M.M. Konstantinov. *Computational Methods for Linear Control Systems.* Prentice-Hall, Hertfordshire, UK, 1991.

[39] L.S. Pontryagin, V. Boltyanskii, R. Gamkrelidze, and E. Mishenko. *The Mathematical Theory of Optimal Processes.* Interscience, New York, 1962.

[40] V. Sima. *Algorithms for Linear-Quadratic Optimization*, volume 200 of *Pure and Applied Mathematics.* Marcel Dekker, Inc., New York, NY, 1996.

[41] P. Van Dooren. A generalized eigenvalue approach for solving Riccati equations. *SIAM J. Sci. Statist. Comput.*, 2:121–135, 1981.

[42] C.F. Van Loan. A symplectic method for approximating all the eigenvalues of a Hamiltonian matrix. *Linear Algebra Appl.*, 61:233–251, 1984.

[43] K. Zhou, J.C. Doyle, and K. Glover. *Robust and Optimal Control.* Prentice-Hall, Upper Saddle River, NJ, 1996.